

# Localized backreacted flavor branes in holographic QCD

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**ABSTRACT:** We investigate the perturbative (in  $g_s N_{D8}$ ) backreaction of localized D8 branes in D4-D8 systems including in particular the Sakai Sugimoto model. We write down the explicit expressions of the backreacted metric, dilaton and RR form. We find that the backreaction remains small up to a radial value of  $u \ll \ell_s/(g_s N_{D8})$ , and that the background functions are smooth except at the D8 sources. In this perturbative window, the original embedding remains a solution to the equations of motion. Furthermore, the fluctuations around the original embedding, describing scalar mesons, do not become tachyonic due to the backreaction in the perturbative regime. This is due to a cancellation between the DBI and CS parts of the D8 brane action in the perturbed background.

**KEYWORDS:** AdS-CFT Correspondence, Intersecting branes models, Gauge-gravity correspondence.

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## 1. Introduction

By now, AdS/CFT has become a standard tool in theoretical physics for the study of gauge theories at strong coupling. In many “stringy” models of gauge dynamics fundamental matter is included by embedding a set of “flavor branes” in addition to the “glue/color branes.” In such a setup, the strings connecting only to the “glue branes” are in the adjoint of the  $U(N_c)$  group, giving gauge particles (multiplets), and those connecting only to the “flavor branes” are in the adjoint of  $U(N_f)$ , giving the mesons (meson multiplets), and those connected to both the “flavor” and “glue” branes are in the fundamental representation

of both the  $U(N_c)$  and  $U(N_f)$  groups, giving the quarks (matter/quark multiplets). Anti-quarks are obviously depicted by similar strings with the opposite orientation.

In principle for large  $N_c$  and large  $N_f$  one could go to a combined near horizon limit, translate the branes into fluxes<sup>1</sup> and derive the gravity background that is a holographic dual of a gauge system with gluons and quarks ( and, if the model is supersymmetric, their supersymmetric partners). However, in practice such models are being constructed using the probe approximation. In this approximation one uses a gravity background built from the near horizon limit of a large  $N_c$  glue branes and adds to it a set of  $N_f$  flavor probe branes. The basic assumption of the probe approximation is that for the case of  $N_f \ll N_c$  the backreaction of the probes on the background can be neglected. The flavor physics is then extracted by analyzing the effective action that describes the flavor branes in the glue background, namely the DBI action plus the CS action. This practice was introduced in [1] in the context of the  $AdS_5 \times S^5$  model, for a confining background in [2] and subsequently in a large variety of other models [3–5]. Exceptions to this are certain fully backreacted non critical models like [6–8], and [9].

A landmark holographic model of chiral symmetry and chiral symmetry breaking is the model of Sakai and Sugimoto [10]. This model is based on the incorporation of a stack of  $N_f$  D8 and  $N_f$  anti- D8 flavor branes into the background of near extremal  $D4$  branes [11]. In the latter background one compactifies one of the world volume coordinates of the D4 branes on a circle of radius  $R$ . For energies  $E \ll 1/R$  the background describes a four dimensional system with gluon degrees of freedom plus contaminating Kaluza Klein modes. The profile of the flavor branes determined by the DBI action is that of a U shape. This provides a simple geometrical picture of chiral symmetry breaking, namely, for large radial direction (see figure 2), which corresponds to the UV limit of the gauge theory, the stack of the D8 branes and of the anti-D8 branes are separated and hence there is a  $U_L(N_f) \times U_R(N_f)$  symmetry, and in the IR limit the two stacks merge one into the other and thus only the diagonal  $U(N_f)$  survives as a symmetry. A variety of physical properties of meson and baryon physics has been extracted from the model. These include the massive meson spectrum, the massless Goldstone pions [10], certain decay rates [12] as well as the thermal behavior of hadrons [13, 14].

The validity of the model [10], is the same as all other probe models:  $N_f \ll N_c$ . To contact to real hadron physics, one obviously is interested in the case where the number of flavors is similar to that of the colors and both are not large. To get down to small  $N_c$  one will have to invoke a full string theory rather than a effective gravity model. However, increasing the ratio of  $N_f/N_c$  can still be done in the context of an effective field theory, provided we go beyond the probe approximation and incorporate the backreaction of the flavor branes on the gravity background. This may enable us to determine the flavor dependence of certain physical properties of the gauge theory which we expect to be  $N_f$  dependent, for example the beta function, or the ratio of viscosity to entropy density of

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<sup>1</sup>Bearing in mind that one must keep the open string spectrum associated with the flavor branes, even though they are producing macroscopic flux. Such a case is a full backreaction, but not a decoupling limit. Recall that global symmetries of the field theory translate into gauge symmetries in the gravity: gauging a large  $N_f$  flavor group using supergravity alone is unfeasible.

the quark-gluon fluid [15].<sup>2</sup>

Similar studies for localized backreactions in D3-D7 systems include [17–21]. In [17], very general framework for studying type IIB supergravity with metric/five form and holomorphic dilaton/axion. The work of [18–20] studied when the D7 branes are located at singular points in manifolds, and [21] studied the form of the solution to the equations following from the D3-D7 system and effects on other probe branes in such backgrounds. All these studies worked using the supergravity alone, while here we will derive delta function source terms from an action of the form

$$S_{\text{Bulk}} + K_8(S_{\text{DBI}} + S_{\text{CS}}). \quad (1.1)$$

We will use this action to determine how to source the bulk fields. Although we obtain the full equations of motion from this, we will study their solutions in a perturbative limit. Therefore, while we are going beyond the probe approximation, we are still confined to the regime  $N_f \ll N_c$  for the simple reason that we want the series in powers of  $N_f/N_c$  to converge quickly.

In fact there is an even more important motivation to explore the model of [10] beyond the probe approximation, and that is the issue of the stability of the model. One may wonder whether the model is stable only in the probe approximation and that the backreaction of the probe branes on the background does not destabilize the setup. A simplified picture of the model is that of the circle of the compactified direction with the two endpoints of the stacks of probe branes and anti branes which can be represented as a  $+N_f$  charge located at one point on the circle and  $-N_f$  charge located at the antipodal point. In this simplified “electrostatic” setup if one of the charges gets a slight perturbation in one direction it will be attracted to the opposite charge and will not be driven back to its original location. Moreover, the antipodal setup described in [10] has been generalized to a family of setups where the separation distance between the brane and anti-brane is taken to be  $L \leq \pi R$ . For these cases the “electrostatic instability” is even more severe. The question is therefore whether this naive intuition is justified and the backreaction of the probe branes indeed destabilizes the model. On the other hand there is a naive argument why the perturbative backreacted system should be stable and non tachyonic. Since the gauge holographic dual of the model before perturbing it has a spectrum with a mass gap ( apart from the pions), a small a small perturbation cannot bridge the gap and produce tachyonic modes [22].

Further, one may wonder what happens to the the dilaton tadpole condition, given that this is a D8  $\bar{D}8$  on a circle, and both branes and anti branes source the dilaton in the same way. Hence, for these codimension one flavor branes one anticipates that the dilaton will have a cusp behavior at the location of the probe branes as well as a cusp (and not an anti-cusp) at the anti-brane. It seems naively that there is no way to sew together these two cusps.

Thus the goal of this paper is to compute the leading order backreacted background, and address the stability, and the dilaton tadpole. To do so, we write down the full action

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<sup>2</sup>In [16], the leading order correction in  $N_f/N_c$  of this property was determined in the context of a model with D7 branes in near extremal  $AdS_5 \times S^5$  background. However, in this model it was shown that the zero mode, which is equivalent to smearing, is all that is necessary to this level in  $N_f/N_c$ .

of the system (of the form in (1.1)) which is composed of the action of massive type  $II_A$  supergravity and the nine dimensional DBI +CS actions associated with the the  $D8$  flavor branes. At this point one often invokes a smearing of the flavor branes along their transverse direction [23–25] which renders the combined action into a ten dimensional one. This approach simplifies the analysis by turning the equations of motion (EOMs) into ordinary differential equations (ODEs) of some radial variable. However, we expect that certain physical questions may not be answered using this procedure, for example the stability discussed above. Thus we avoid using the smearing technique and we keep the flavor branes as localized objects. This yields delta function source terms for the equations of motion of the graviton, dilaton and the  $F_{(10)}$  RR field strength form associated with the  $D8$  branes. The complexity of these equations is increased, relative to the smearing technique, because the EOMs are now partial differential equations (PDEs); the relevant functions must depend on the coordinate(s) transverse to the flavor brane.<sup>3</sup>

We solve these equations perturbatively to the leading order in  $N_f/N_c$ . We take 3 cases for the background to help address the questions in stages, and gain intuition for how the solutions should behave.

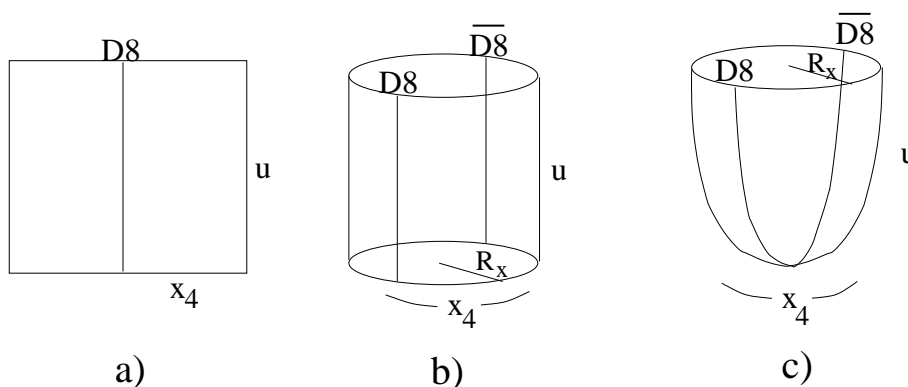
We first solve for the simplified system of a decompactified transverse coordinate of the  $D8$  branes, which has been studied in its own right in [26, 27](see figure 1 (a)). For this case we were able to find exact solutions of the partial differential equations. To our surprise we have found that whereas the solutions for the perturbations of some background fields behave, as we have expected, with a cusp at the location of the probe branes (a “ $\Lambda$ ” shape), for other the behavior is of an inverted cusp dressed with a double hump structure (an “m”, see figure 3). We explore the ranges where we expect the supergravity to be a good description and find that  $u \gg \ell_s^4/R_{D4}^3$ , and that the perturbative results are good up to  $u \ll 1/Q_f \equiv 4\pi\ell_s/(g_s N_f)$ . Further, from this study, it becomes plausible that compactifying the  $x_4$  direction is possible, as the functions die off at large  $x_4$ .

Next, we address the compactified (but extremal) case both to view the effects of compactification, but also as a rough approximation of the “cigar” case at large  $u$  (see figure 1 (b)). We first treat this case where we sum over the images of the uncompactified case, and then as a Fourier decomposition. From this we see that there is no issue with the dilaton tadpole constraint: the two cusps meets smoothly. We find that the decompactified limit emerges at large  $u$ . Both methods are applied because, although both series always converge, one series converges very quickly at large  $u$ , while the other converges quickly at small  $u$ .

Finally we analyze the system of the near extremal  $D4$  branes. We find that in this case the perturbation theory is good for  $u \ll 1/Q_f$  which is generically stronger than the  $u^3 \ll R_{D4}^3/g_s^4$  supergravity regime. Further, we find that the supergravity description is valid near  $u = U_K$ , with the additional constraint  $(U_K/R_{D4})^{3/4}Q_f \ll 1/\ell_s$ . This translates into the requirement that  $(T_{st}/M_{gb}^2)^{1/2}\lambda_4 N_f/N_c \ll 1$  where  $T_{st}$  is the string tension,  $M_{gb}$  is the typical glueball mass, and  $\lambda_4 = g_4^2 N_c$  is the 't Hooft coupling. For large values of the

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<sup>3</sup>For this added complication, though, we simplify the equations by using a perturbative approach. In some sense, this is complimentary to smearing: one smears the branes to obtain non-linear ODEs to solve; we instead perturb the equations to obtain linear PDEs.



**Figure 1:** The three cases studied: a) the uncompactified  $x_4$  case, b) the compactified  $x_4$  case, and c) the near extremal “cigar” case.

radial direction the solution is obviously like that of the extremal compactified case. We use a Fourier decomposition to show that a finite expansion around the tip of the cigar is possible, and then implement this expansion for the first few terms.

Once we have established the perturbative solutions, we proceed to analyze the stability of the system. We first show that the solution of the embedding of the flavor brane at the probe level persists also in the leading order backreaction. We further show that the fluctuations of the embedding, which correspond to scalar mesons in the dual gauge theory, are non tachyonic. Hence we shown that the system is stable at least for an action that is quadratic in the fluctuations. This is due to a cancelation between the electrostatic repulsion (CS action) and the gravitational attraction (DBI action). Hence, the above analogy with an electrostatic problem is not quite justified: the electric repulsion is canceled by a gravitational attraction. The only other force is that of the tension of the brane, which is restorative. The corrections to this force, while interesting, cannot change the qualitative feature of stability while the perturbative analysis is valid (however, the effects of the non-perturbative backreaction is still an open question).

The paper is organized as follows. In the next section (2) we briefly review the general setup of the problem, namely, the Sakai Sugimoto [10] model and the massive type  $II_A$  supergravity action [28, 29]. In section 3 we write the supergravity EOMs that incorporate the backreaction of the probe branes. We then introduce an ansatz for the metric which we substitute into the equations. The perturbative parameter is defined, and these equations are expanded. The gauge invariance, in the form of small coordinate transformations, of the system is discussed. In section 4 we present the solutions of the backreacted EOM. We start with the solutions for the uncompactified case, and then by summing over images the solutions for the compactified extremal case is constructed. We also use Fourier analysis to study this case. This enables us to determine the UV behavior of the near extremal case because the geometries are identical at large radius. The third step is to write down the solutions for the near extremal case in the region close to the horizon. In the following

section we analyze the stability of the system. We first show that the solution of the EOMs that follow from the backreacted DBI+CS action are the same as those of the unperturbed solution. Finally we shown that the spectrum of fluctuations around this embedding is tachyon free and hence we conclude that to the leading order in  $N_f/N_c$  the system is stable.

## 2. Brief review of the general setup

Before we start the analysis of the backreaction of the flavor branes, we briefly review the two main ingredients of the general setup of the problem, namely, the Sakai Sugimoto model and the action of the massive type IIA supergravity. The reader familiar with these topics should skip to the next section.

### 2.1 Sakai-Sugimoto (SS) model

Constructing holographic models duals of gauge dynamics that admits confinement is by now a relatively easy task. Incorporating flavor chiral symmetry, on the other hand, turns out to be more complicated. A prototype model that includes both phenomena is the Sakai-Sugimoto model [10]. It is a model of a holographic dual for a 3 + 1 dimensional gauge theory with a continuous  $SU(N_f) \times SU(N_f)$  flavor chiral symmetry which is spontaneously broken. It is based on the incorporation of  $N_f$  D8-branes and  $N_f$  anti-D8-branes into Witten’s model [11]. The latter describes the near horizon limit of  $N_c$  D4-branes, compactified on a circle of radius  $R$  ( $x_4 \equiv x_4 + 2\pi R$ ) with anti-periodic boundary conditions for the fermions. The D8-branes are placed at  $x_4 = 0$  and the anti-D8-branes at  $x_4 = L$ . The gauge theory dual of this SUGRA setup is a 4 + 1 dimensional  $SU(N_c)$  maximally supersymmetric gauge theory, compactified on a circle with anti-periodic boundary conditions for the adjoint fermions, and coupled to  $N_f$  left-handed fermions in the fundamental representation of  $SU(N_c)$  localized at  $x_4 = 0$ , and to  $N_f$  right-handed fermions in the fundamental representation localized at  $x_4 = L$ .

The basic assumption of the model is that in the limit of  $N_f \ll N_c$  one can ignore the back-reaction of the  $N_f$  D8 branes and  $N_f$  anti-D8 branes. As mentioned above the goal of the present work is to examine in details the back-reaction of the D8 and anti-D8 on the background. With the probe assumption the closed type IIA string background is given by:

$$\begin{aligned}
 ds^2 &= \left(\frac{u}{R_{D4}}\right)^{3/2} \left[-dt^2 + \delta_{ij} dx^i dx^j + f(u) dx_4^2\right] + \left(\frac{R_{D4}}{u}\right)^{3/2} \left[\frac{du^2}{f(u)} + u^2 d\Omega_4^2\right], \\
 F_{(4)} &= \frac{3R_{D4}^3}{g_s} \Omega_4, \quad e^\phi = g_s \left(\frac{u}{R_{D4}}\right)^{3/4}, \quad R_{D4}^3 \equiv \pi g_s N_c l_s^3, \quad f(u) \equiv 1 - \left(\frac{U_K}{u}\right)^3,
 \end{aligned}
 \tag{2.1}$$

where  $t$  is the time direction and  $x^i$  ( $i = 1, 2, 3$ ) are the uncompactified world-volume coordinates of the D4 branes,  $x_4$  is a compactified direction of the D4-brane world-volume which is transverse to the probe D8 branes,  $d\Omega_4^2$  is the metric of a unit four-sphere and  $\epsilon_4$  is its volume form, and  $g_s$  is related to the 4 + 1 dimensional gauge coupling by  $g_5^2 = (2\pi)^2 g_s l_s$ . The submanifold spanned by  $x_4$  and  $u$  has the topology of a cigar with  $u \geq U_K$ , and

requiring that this has a non-singular geometry gives a relation between  $U_K$  and  $R_x$ ,

$$R_x = \frac{2}{3} \left( \frac{R_{D4}^3}{U_K} \right)^{1/2}. \quad (2.2)$$

The parameters of this gauge theory, the five-dimensional gauge coupling  $g_5$ , the low-energy four-dimensional gauge coupling  $g_4$ , the glueball mass scale  $M_{\text{gb}}$ , and the string tension  $T_{\text{st}}$  are determined from the background (2.1) in the following form:

$$g_5^2 = (2\pi)^2 g_s l_s, \quad g_4^2 = \frac{g_5^2}{2\pi R_x} = 3\sqrt{\pi} \left( \frac{g_s U_K}{N_c l_s} \right)^{1/2}, \quad M_{\text{gb}} = \frac{1}{R_x},$$

$$T_{\text{st}} = \frac{1}{2\pi l_s^2} \sqrt{g_{tt} g_{xx}}|_{u=U_K} = \frac{1}{2\pi l_s^2} \left( \frac{U_K}{R_{D4}} \right)^{3/2} = \frac{2}{27\pi} \frac{g_4^2 N_c}{R_x^2} = \frac{\lambda_5}{27\pi^2 R_x^3}, \quad (2.3)$$

where  $\lambda_5 \equiv g_5^2 N_c$ ,  $M_{\text{gb}}$  is the typical scale of the glueball masses computed from the spectrum of excitations around (2.1), and  $T_{\text{st}}$  is the confining string tension in this model (given by the tension of a fundamental string stretched at  $u = u_K$  where its energy is minimized). The gravity approximation is valid whenever  $\lambda_5 \gg R_x$ , otherwise the curvature at  $u \sim U_K$  becomes large. Note that as usual in gravity approximations of confining gauge theories, the string tension is much larger than the glueball mass scale in this limit. At very large values of  $u$  the dilaton becomes large, but this happens at values which are of order  $N_c^{4/3}$  (in the large  $N_c$  limit with fixed  $\lambda_5$ ), so this will play no role in the large  $N_c$  limit that we will be interested in. The Wilson line of this gauge theory (before putting in the D8-branes) admits an area law behavior [30], as can be easily seen using the conditions for confinement of [31].

The gauge theory dual to the SUGRA background (2.1) is in fact not four dimensional even at energies lower than the Kaluza-Klein scale  $1/R_x$  since the masses of the glueballs are also  $M_{\text{gb}} = 1/R_x$ , namely, there is no real separation between the confined four-dimensional fields and the higher Kaluza-Klein modes on the circle. As discussed in [11], in the opposite limit of  $\lambda_5 \ll R_x$ , the theory approaches the  $3+1$  dimensional pure Yang-Mills theory at energies small compared to  $1/R_x$ , since in this limit the scale of the mass gap is exponentially small compared to  $1/R$ .

The probe flavor D8-branes span the coordinates  $t, x^i, \Omega_4$ , and trace a curve  $u(x_4)$  in the  $(x_4, u)$ -plane. Near the boundary at  $u \rightarrow \infty$  we want to have  $N_f$  D8-branes localized at  $x_4 = 0$  and  $N_f$  anti-D8-branes (or D8-branes with an opposite orientation) localized at  $x_4 = L$ . Naively one might think that the D8-branes and anti-D8-branes would go into the interior of the space and stay disconnected; however, these 8-branes do not have anywhere to end in the background (2.1), so the form of  $u(x_4)$  must be such that the D8-branes smoothly connect to the anti-D8-branes (namely,  $u$  must go to infinity at  $x_4 = 0$  and at  $x_4 = L$ , and  $du/dx_4$  must vanish at some minimal  $u$  coordinate  $u = u_0$ ). Such a configuration spontaneously breaks the chiral symmetry from the symmetry group which is visible at large  $u$ ,  $U(N_f)_L \times U(N_f)_R$ , to the diagonal  $U(N_f)$  symmetry. Thus, in this configuration the topology forces a breaking of the chiral symmetry.

To determine the profile of flavor probe branes, one has to solve the equations of motion of that follow from the DBI + CS action that describes the probe branes. It is easy to



check that the CS term in the D8-brane action does not affect the solution of the equations of motion. More precisely, the equation of motion of the gauge field has a classical solution of a vanishing gauge field, since the CS term includes terms of the form  $C_5 \wedge F \wedge F$  and  $C_3 \wedge F \wedge F \wedge F$ . So, we are left only with the DBI action. The induced metric on the D8-branes is

$$ds_{D8}^2 = \left(\frac{u}{R_{D4}}\right)^{3/2} [-dt^2 + \delta_{ij} dx^i dx^j] + \left(\frac{u}{R_{D4}}\right)^{3/2} \left[ f(u) + \left(\frac{R_{D4}}{u}\right)^3 \frac{u'^2}{f(u)} \right] dx_4^2 + \left(\frac{R_{D4}}{u}\right)^{3/2} u^2 d\Omega_4^2 \tag{2.4}$$

where  $u' = du/dx_4$ . Substituting the determinant of the induced metric and the dilaton into the DBI action, we obtain (ignoring the factor of  $N_f$  which multiplies all the D8-brane actions that we will write):

$$S_{\text{DBI}} = T_8 \int dt d^3 x dx_4 d^4 \Omega e^{-\phi} \sqrt{-\det(\hat{g})} = \frac{\hat{T}_8}{g_s} \int dx_4 u^4 \sqrt{f(u) + \left(\frac{R_{D4}}{u}\right)^3 \frac{u'^2}{f(u)}}, \tag{2.5}$$

where  $\hat{g}$  is the induced metric (2.4) and  $\hat{T}_8$  includes the outcome of the integration over all the coordinates apart from  $dx_4$ . The simplest way to solve the equation of motion is by using the Hamiltonian of the action (2.5), which is conserved (independent of  $x_4$ ):

$$\frac{u^4 f(u)}{\sqrt{f(u) + \left(\frac{R_{D4}}{u}\right)^3 \frac{u'^2}{f(u)}}} = \text{constant} = u_0^4 \sqrt{f(u_0)}, \tag{2.6}$$

where on the right-hand side of the equation we assumed that there is a point  $u_0$  where the profile  $u(x_4)$  of the brane has a minimum,  $u'(u = u_0) = 0$ .<sup>4</sup> We need to find the solution in which as  $u$  goes to infinity,  $x_4$  goes to the values  $x_4 = 0, L$ ; this implies

$$\int dx_4 = 2 \int \frac{du}{u'} = L \tag{2.7}$$

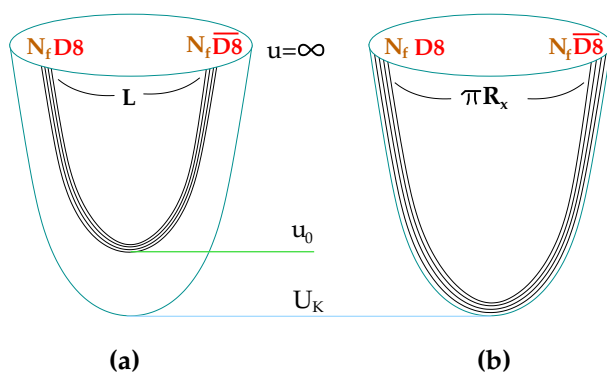
with  $u'$  given (as a function of  $u$ ) by (2.6) (note that  $u$  is a double-valued function of  $x_4$  in these configurations, leading to the factor of two in (2.7)). The form of this profile of the D8 brane is drawn in figure 2(a). Plugging in the value of  $u'$  from (2.6) we find

$$L = \int dx_4 = 2 \int_{u_0}^{\infty} \frac{du}{u'} = 2 R_{D4}^{3/2} \int_{u_0}^{\infty} du \frac{1}{f(u) u^{3/2} \sqrt{\frac{f(u) u^8}{f(u_0) u_0^8} - 1}} = \frac{2}{3} \left(\frac{R_{D4}^3}{u_0}\right)^{1/2} \sqrt{1 - y_K^3} \int_0^1 dz \frac{z^{1/2}}{(1 - y_K^3 z) \sqrt{1 - y_K^3 z - (1 - y_K^3) z^{8/3}}}, \tag{2.8}$$

where  $y_K \equiv u_K/u_0$ . Small values of  $L$  correspond to large values of  $u_0$ . In this limit we have  $y_K \ll 1$  leading to  $L \propto \sqrt{R_{D4}^3/u_0}$ . For general values of  $L$  the dependence of  $u_0$  on  $L$  is more complicated.

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<sup>4</sup>This type of analysis was done previously for Wilson line configurations. See, for instance, [30].



**Figure 2:** The dominant configurations of the D8 and anti-D8 probe branes in the Sakai-Sugimoto model at zero temperature, which break the chiral symmetry. The same configurations will turn out to be relevant also at low temperatures. On the left a generic configuration with an asymptotic separation of  $L$ , that stretches down to a minimum at  $u = u_0$ , is drawn. The figure on the right describes the limiting antipodal case  $L = \pi R_x$ , where the branes connect at  $u_0 = U_K$ .

There is a simple special case of the above solutions, which occurs when  $L = \pi R_x$ , namely the D8-branes and anti-D8-branes lie at antipodal points of the circle. In this case the solution for the branes is simply  $x_4(u) = 0$  and  $x_4(u) = L = \pi R_x$ , with the two branches meeting smoothly at the minimal value  $u = u_0 = U_K$  to join the D8-branes and the anti-D8-branes together. This type of antipodal solution is drawn in figure 2(b). It was shown in [10] that this classical configuration is stable, by analyzing small fluctuations around this configuration and checking that the energy density associated with them is non-negative.

In general for  $L < \pi R_x$ , there is a family of smooth configurations characterized by  $L$  or by the minimal value of  $u$ ,  $u_0$ . This class of solution is shown in 2(a)

The Sakai-Sugimoto model has 3 dimensionful parameters:  $\lambda_5$ ,  $L$  and  $R_x$ , and gravity is reliable whenever  $\lambda_5 \gg R_x$ . The physics depends on the two dimensionless ratios of these two parameters; In the gravity limit the mass of the (low-spin) mesons is related to  $1/L$  [13] while the mass of the (low-spin) glueballs is related to  $1/R_x$ . As discussed above, in the limit  $\lambda_5 \ll R_x$  this theory approaches (large  $N_c$ ) QCD at low energies. This remains true also after adding the flavors, at least when  $L$  is of order  $R_x$ .

The thermal phases of the model were analyzed in [13, 14]. The back-reaction of the flavor brane at non zero temperature is not addressed in the present paper and will be described in a future publication.

## 2.2 Massive type IIA and 8 branes

One expects  $p + 1$  dimensional objects to naturally couple to a  $p + 1$  form potential. Therefore, one expects a D8 brane to couple to a nine form potential. Conventional type IIA supergravity has no such form, and so some modification of the theory is necessary to describe the backreaction of D8 branes. This extension was first found by Romans [28], and then further generalized to admit localized D8 solutions in [29]. The relevant kappa

symmetric worldvolume actions were constructed in [32]. Further studies of D8 (-Dp) brane backgrounds (systems) are discussed in [33–38].

There exists another massive type IIA, constructed in [39]. This and the Romans’ type IIA were shown to be the only “Higgs type” supersymmetric extensions of massless type IIA in [40], although a third was suggested in [39]. The massive type IIA given in [39] does not admit localized supersymmetric eight-branes, and so we restrict our attention to the theory of Romans [28, 29] which we simply refer to as massive type IIA.

The bosonic part of the action of the massive type IIA supergravity takes the form<sup>5</sup>

$$\begin{aligned}
 S_{IIA_M} &= S_{\text{NS}} + S_R + S_{\text{CS}} + S_M \\
 S_{\text{NS}} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} e^{-2\phi} \left( R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{2 \cdot 3!} H_3 \cdot H_3 \right) \\
 S_R &= -\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left( \frac{1}{2 \cdot 2!} \tilde{F}_2 \cdot \tilde{F}_2 + \frac{1}{2 \cdot 4!} \tilde{F}_4 \cdot \tilde{F}_4 \right) \\
 S_{\text{CS}} &= -\frac{1}{2\kappa_{10}^2} \int d^{10}x \frac{1}{2} \frac{1}{2! \cdot 4! \cdot 4!} \epsilon^{\mu_1 \dots \mu_{10}} B_{\mu_1 \mu_2} \hat{F}_{\mu_3 \dots \mu_6} \hat{F}_{\mu_7 \dots \mu_{10}} \\
 S_M &= -\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \frac{1}{2} M^2 + \frac{1}{2\kappa_{10}^2} \int M F_{10}
 \end{aligned} \tag{2.9}$$

where  $\cdot$  denotes contraction of indices with inverse metrics, and  $\epsilon$  is antisymmetric in indices and takes values  $\pm 1$ . In the above action  $F_{n+1} = dA_n$  and

$$\begin{aligned}
 \tilde{F}_2 &= F_2 + M B_2 \\
 \hat{F}_4 &= F_4 + \frac{1}{2} M B_2 \wedge B_2 \\
 \tilde{F}_4 &= F_4 - A_1 \wedge H_3 + \frac{1}{2} M B_2 \wedge B_2.
 \end{aligned} \tag{2.10}$$

One notes that from the above definitions,  $F_2$  may be absorbed completely by a shift in  $B_2$ , but only when  $M \neq 0$ . One views this as a “Higgsing” where the degrees of freedom associated with  $F_2$  become the longitudinal modes of a massive  $B_2$ . The equation of motion for  $F_2$ , therefore, must only be imposed in the massless limit. In appendix A, we include the equation of motion associated with  $A_1$  so that an  $M \rightarrow 0$  limit is clear. In the next section we turn to including sources in the action.

### 3. Backreaction of D8 branes

In this section, we will find the equations of motion that govern the D4-D8 systems of interest, including the contribution from the DBI + CS brane action. We present our ansatz, and the perturbative parameter we will use to linearize the equations, and finally present the separated linearized equations. Further, we find the small coordinate transformations that leave the form of our ansatz unchanged (to the order we are working): these are gauge transformations of the linearized equations.

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<sup>5</sup>We use the notation of chapter 12 of Polchinski [41]

### 3.1 Finding the equations: ansatz and separation

For the remainder of the paper, we will be concerned with D4-D8 systems, and because neither of these branes source (directly) either  $A_\mu$  or  $B_{\mu\nu}$ , we set them to zero. After truncation to the  $A_\mu = 0, B_{\mu\nu} = 0$  sector, the equations of motion for the type IIA massive supergravity are the following:

$$\begin{aligned}
 R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi &= \frac{e^{2\phi}}{2!4!} \left( 4F_\mu^{\alpha_2\alpha_3\alpha_4} F_{\nu\alpha_2\alpha_3\alpha_4} - \frac{1}{2} g_{\mu\nu} (F_{(4)})^2 \right) - \frac{e^{2\phi}}{4} g_{\mu\nu} M^2 \\
 g^{\mu\nu} \nabla_\mu \nabla_\nu \phi &= g^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu \phi) - \frac{1}{4} R \\
 \nabla_{\alpha_1} F^{\alpha_1\alpha_2\alpha_3\alpha_4} &= 0 \\
 \partial_\mu M &= 0 \\
 M &= *F_{(10)} \\
 0 &= \frac{1}{\sqrt{-g}} e^{\beta_1\beta_2\alpha_1\alpha_2\alpha_3\alpha_4\rho_1\rho_2\rho_3\rho_4} F_{\alpha_1\alpha_2\alpha_3\alpha_4} F_{\rho_1\rho_2\rho_3\rho_4}.
 \end{aligned} \tag{3.1}$$

Note that the equation of motion for  $A_1$  in the appendix is trivially satisfied. Again, one must only impose it's equation of motion in the massless limit. However, the equation of motion for  $B_{\mu\nu}$  imposes a constraint, arising from the Chern Simons term  $B \wedge F_4 \wedge F_4$ , on the four form (the last of the above equations). This constraint is easily satisfied for simple 4-form field strengths. Also, note that we have used the dilaton equation of motion (EOM) to eliminate  $R$  from the Einstein equation. This will be important below when we derive the equations when a brane source is present.

We now turn to the modification of the equations of motion (3.1) by adding

$$-K_p \left( \int d^{p+1} \xi e^{-\phi} \sqrt{-g_p} + \int A_{p+1} \right) \tag{3.2}$$

to the action. Here we use  $g_p$  to denote the pullback metric on the  $p + 1$  dimensional submanifold defined by  $X^\alpha(\xi)$ ,

$$(g_p)_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} g_{\mu\nu} \tag{3.3}$$

and  $K$  is the appropriate constant involving the p-brane tension. In this action, we assume that it is consistent to set the world volume U(1) gauge field to zero, as well as ignoring any additional Chern Simons terms (which is appropriate for the cases we wish to consider).

There are two types of fields in this action: those that represent open string degrees of freedom (e.g.  $X^\alpha(\xi)$ ); and those representing closed string degrees of freedom (e.g.  $g_{\alpha\beta}$ ). Of course when varying with respect to the closed string degrees of freedom, 10D delta functions appear, of which  $p + 1$  are integrated leaving behind a  $(10 - (p + 1))$  dimensional delta function source term, as we should expect. Varying the above action with respect to the  $p + 1$  form potential adds a delta function source to the form fields equation of motion of the generic form

$$\frac{1}{(2\kappa_{10}^2)} (d * F) - K_p \delta^{10-p-1} \frac{(\epsilon_{10} \cdot \epsilon_{p+1})}{(p + 1)!} = 0. \tag{3.4}$$

Note that the product  $\epsilon_{10} \cdot \epsilon_{p+1}$  is sensitive to the orientation of the submanifold defined by  $X^\alpha$ . For example, in the case of the SS model with the antipodal embedding of the D8 branes, there is both a positive delta function and a negative delta function in  $x_4$  accounting for the orientation reversal of the brane (it is oriented in the  $\pm u$  direction). For D8 branes, however, one should replace  $d * F$  with  $dM$  because  $M$  is the term that appears with  $F_{10}$  in the action. Hence,  $M$  is in fact piecewise constant in backgrounds with localized D8, as we will see below.

We restrict ourselves to embedding functions of the form  $X^a(\xi) = \xi^a$  and the remaining  $X^i$  are arbitrary constants. Varying the full action with respect to the dilaton and graviton is now straightforward, and the equations of motion are

$$\begin{aligned}
 R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi - \frac{e^{2\phi}}{2!4!} \left( 4F_\mu^{\alpha_2\alpha_3\alpha_4} F_{\nu\alpha_2\alpha_3\alpha_4} - \frac{1}{2} g_{\mu\nu} (F_{(4)})^2 \right) + \frac{e^{2\phi}}{4} g_{\mu\nu} M^2 & \quad (3.5) \\
 + \frac{K_8 2\kappa_{10}^2}{2} e^\phi \frac{\sqrt{-g_p} |_{\xi^i=x^i}}{\sqrt{-g}} \left( g_p^{ab} \frac{\partial X^\alpha}{\partial \xi^a} \frac{\partial X^\beta}{\partial \xi^b} \Big|_{\xi^i=x^i} g_{\alpha\mu} g_{\beta\nu} - \frac{1}{2} g_{\mu\nu} \right) \delta(x^\alpha - X^\alpha(x)) = 0 \\
 R + 4g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - 4g^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu \phi) - \frac{K_8 2\kappa_{10}^2}{2} e^\phi \frac{\sqrt{-g_p} |_{\xi^i=x^i}}{\sqrt{-g}} \delta(x^\alpha - X^\alpha(x)) = 0 \\
 \frac{1}{(2\kappa_{10}^2)} dM - K_8 \delta(x^\alpha - X^\alpha(x)) \frac{(\epsilon_{10} \cdot \epsilon_9)}{9!} = 0 \\
 *F_{(10)} = M.
 \end{aligned}$$

with other equations of motion left unchanged. The delta functions appearing above may be simplified by taking them to be functions only of  $x_4$ ,  $\delta(x^\alpha - X^\alpha(x)) \equiv \Delta(x_4)$ , which is appropriate for the antipodal embedding in the Sakai Sugimoto model. As expected, only the RR couplings to the branes are sensitive to the orientation of the branes. Further, the epsilons appearing above take values  $\pm 1$ , and do not contain factors of  $\sqrt{-g}$ .<sup>6</sup>

The tensor structure of the Einstein equations can be easily read: the delta function strength is proportional to the metric and dilaton, and comes with a + sign for a direction along the D8 brane, and comes with a - sign for those not along the D8 brane. Although above we have written the effect of a D8 brane, the above arguments work also for an arbitrary  $p$  brane: it simply changes which RR form field equation of motion gets the delta function source, and how many directions of the Einstein's equations get (-) vs. (+) delta functions.

For the remainder of this work, we will take the solution to the  $M$  and  $F_{10}$  equations

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<sup>6</sup>As a simple check of the above signs and numerical factors, one can simply check the following. The coefficient in front of  $g_p^{ab} \partial_a X^\alpha \partial_b X^\beta g_{\alpha\mu} g_{\beta\nu}$  can be checked against that of  $R_{\mu\nu}$ . Before using the dilaton equation of motion, the Einstein equations contain  $-\frac{1}{2} R g_{\mu\nu}$ , and this coefficient must match that of  $g_p^{ab} \partial_a X^\alpha \partial_b X^\beta g_{\alpha\mu} g_{\beta\nu}$ , except that one multiplies the latter by an additional  $-K_8 2\kappa_{10}^2 \sqrt{-g_p} / \sqrt{-g}$ . This is because they are obtained from similar terms in the action,  $\frac{1}{2\kappa_{10}^2} \sqrt{-g} e^{-2\phi} R$  and  $-K_8 e^{-\phi} \sqrt{-g_p}$  respectively. The factor in the dilaton equation is also obtained similarly, as the coefficient of  $R$  is obtained by varying  $\frac{1}{2\kappa_{10}^2} \sqrt{-g} e^{-2\phi}$  w.r.t.  $\phi$  and the delta function coefficient is obtained by varying  $-K_8 \sqrt{-g} e^{-\phi}$ , and so a factor of 1/2 arises. The second part of the delta function term in Einstein's equations is similarly found by checking that one adds  $\frac{1}{2} g_{\mu\nu}$  times that of the dilaton term.

of motion to be

$$\begin{aligned}
 M &= \pm 2\kappa_{10}^2 K_8/2 = \pm \frac{N_f}{4\pi\ell_s} \equiv \pm \frac{Q_f}{g_s} \\
 *F_{10} &= M
 \end{aligned}
 \tag{3.6}$$

where the + is used on one side of the D8, and – is used on the other.<sup>7</sup>

The remainder of the paper will be devoted to solving the remaining equations of motion perturbatively, and the perturbative control parameter will be explained shortly. However, at this point an important note is in order: delta functions in codimension  $10 - (p + 1) \neq 1$  have singularities at the location of the delta function. Codimension one is special in that the Green’s function is of the form  $|x|$ ,<sup>8</sup> and hence is finite at the source. We therefore expect that the perturbative approach is most natural for D8 branes, as the back reaction can be made small, even close to the brane. Hence, the perturbative approach that we take may not be as natural for higher codimension branes.

To define the small parameter in our expansion, we make the following observations. Given the solution of the  $M, F_{10}$  sector, the new (relative to the massless IIA equations) terms in the equations of motion come with the powers of  $2\kappa_{10}^2 K_8 e^\phi \sim g_s N_f$ . This is what we shall use as a control parameter for our perturbative expansion. From now on, we will simply take  $2\kappa_{10}^2 K_8/2 \equiv Q_f/g_s$  as the definition of our small parameter  $Q_f$ . Another way of phrasing this is that in the holographic limit, there is a scale  $R_c$  such that  $g_s \sim (R_c/l_s)/N_c$  where  $R_c/l_s \gg 1$  is large but held fixed. Hence, one may view our perturbation as limit on  $g_s N_f \sim N_f/N_c$ , which is the basis for the probe approximation.

To solve the equations, we will still need to take an ansatz, and we motivate it as follows. The eight brane doesn’t directly couple to  $F_4$ , and further the  $SO(5)$  symmetry of the 4 sphere is not broken for this brane configuration. Hence, we take that there is no change in  $F_4$  to leading order in  $Q_f$ . Further, because we take the solution  $M \propto \pm Q_f/g_s$ , so that in the Einstein equations, the term  $g_{\mu\nu} M^2 e^{2\phi} \sim \mathcal{O}(Q_f^2)$  may be ignored.

Therefore, we assume that only metric and dilaton perturbations are necessary, so we take a general ansatz of the form

$$\begin{aligned}
 ds^2 &= e^{2A(u,x_4)} (-dt^2 + dx_i^2) + e^{2B(u,x_4)} dx_4^2 + e^{2G(u,x_4)} du^2 + e^{2C(u,x_4)} d\Omega_4^2 \\
 \phi(u, x_4) &= \frac{1}{2} \hat{\phi}(u, x_4) + 2A(u, x_4) + 2C(u, x_4) \\
 F_{(4)} &= Q_c \Omega_4
 \end{aligned}
 \tag{3.7}$$

where  $Q_c = 3R_4^3/g_s = 3\pi N_c l_s^3$ , and  $\Omega_4$  is the volume form of the unit four sphere. It is clear that the  $F_4$  equations are trivially satisfied:  $dF_4 = 0$  because  $\Omega_4$  is closed, and  $d * F_4 = 0$  because  $*F_4$  is some function of  $x_4$  and  $u$  times  $du \wedge dx_4 \wedge dx_i$ , and is therefore closed.

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<sup>7</sup>This terminology only makes sense for branes of dimension  $D - 1$ , as such branes split the space into disjoint regions.

<sup>8</sup>we refer to this behavior as a cusp

In the above, we will expand the above functions as

$$\begin{aligned}
 A(u, x_4) &= A_0(u) + Q_f A_1(u, x_4) & B(u, x_4) &= B_0(u) + Q_f B_1(u, x_4) \\
 C(u, x_4) &= C_0(u) + Q_f C_1(u, x_4) & G(u, x_4) &= G_0(u) + Q_f G_1(u, x_4) \\
 \hat{\phi}(u, x_4) &= \hat{\phi}_0(u) + Q_f \hat{\phi}_1(u, x_4)
 \end{aligned} \tag{3.8}$$

where the 0 subscripted functions are solutions of the  $Q_f = 0$  equations. We linearize and explain how to separate them for the Sakai Sugimoto model in appendix B, and summarize the results here. One must solve the decoupled system

$$\begin{aligned}
 3\partial_u^2 F_1 + \frac{3(4u^3 - U_K^3)\partial_u F_1}{u(u^3 - U_K^3)} + \frac{g_s Q_c u^3 \partial_{x_4}^2 F_1}{(u^3 - U_K^3)^2} \\
 - \frac{54uF_1}{(u^3 - U_K^3)} + \frac{2u(Q_c g_s)^{\frac{2}{3}} \sqrt{3}}{(u^3 - U_K^3)} \sqrt{\frac{u}{(Q_c g_s)^{\frac{1}{3}} \left(1 - \frac{U_K^3}{u^3}\right)}} \Delta = 0
 \end{aligned} \tag{3.9}$$

$$\begin{aligned}
 3\partial_u^2 F_2 + \frac{3(4u^3 - U_K^3)\partial_u F_2}{u(u^3 - U_K^3)} + \frac{g_s Q_c u^3 \partial_{x_4}^2 F_2}{(u^3 - U_K^3)^2} \\
 - \frac{4u(Q_c g_s)^{\frac{2}{3}} \sqrt{3}}{(u^3 - U_K^3)} \sqrt{\frac{u}{(Q_c g_s)^{\frac{1}{3}} \left(1 - \frac{U_K^3}{u^3}\right)}} \Delta = 0
 \end{aligned} \tag{3.10}$$

$$\begin{aligned}
 -4\partial_u^2 \hat{\phi}_1 - \frac{4}{3} \frac{u^3 Q_c g_s \partial_{x_4}^2 \hat{\phi}_1}{(u^3 - U_K^3)^2} - \frac{2(u^3 - 7U_K^3)\partial_u \hat{\phi}_1}{u(u^3 - U_K^3)} - \frac{36uU_K^3 \hat{\phi}_1}{(u^3 - U_K^3)^2} \\
 + \frac{\frac{4}{3}u(Q_c g_s)^{\frac{2}{3}} \sqrt{3}}{(u^3 - U_K^3)} \sqrt{\frac{u}{(Q_c g_s)^{\frac{1}{3}} \left(1 - \frac{U_K^3}{u^3}\right)}} \Delta = 0
 \end{aligned} \tag{3.11}$$

where

$$\Delta = \begin{cases} \delta(x_4) + \delta(x_4 - \pi R_x) & \text{if } x_4 = x_4 + 2\pi R_x \\ \delta(x_4) & \text{if } x_4 \text{ non-compact} \end{cases} \tag{3.12}$$

and then identify the physical degrees of freedom

$$\begin{aligned}
 A_1 &= -\frac{1}{5}F_1 + \frac{1}{10}F_2 - \frac{3}{10}\hat{\phi}_1 \\
 C_1 &= \frac{1}{10}F_1 + \frac{1}{5}F_2 - \frac{1}{10}\hat{\phi}_1 \\
 B_1 = G_1 &= -\frac{1}{5}F_1(u, x_4) + \frac{1}{5}F_2(u, x_4) - \frac{1}{10}\hat{\phi}_1(u, x_4) \\
 &\quad - \frac{2}{5}u\partial_u \hat{\phi}_1(u, x_4) + \frac{3}{5} \frac{\hat{\phi}_1(u, x_4)U_K^3}{(u^3 - U_K^3)} \\
 \phi_1 &= \frac{1}{2}\hat{\phi}_1 + 2A_1 + 2C_1.
 \end{aligned} \tag{3.13}$$

Above, we have made the obvious notation that  $\phi_1$  is the first order correction to the *physical* dilaton. One may read off the combined solution by plugging in these to the equations (B.7) in the appendix.

### 3.2 Gauge freedom

Here we identify the gauge (coordinate transformation) freedom as those transformations in  $u$  and  $x_4$  that leave the metric diagonal (preserves the form of our ansatz). Indeed,

$$\begin{aligned}
 \hat{\phi}_1 &\rightarrow \hat{\phi}_1 - \frac{5}{2} \frac{\lambda(u, x_4)}{u} \\
 A_1 &\rightarrow A_1 + \frac{3}{4} \frac{\lambda(u, x_4)}{u} \\
 B_1 &\rightarrow B_1 + \frac{3}{4} \frac{\lambda(u, x_4)}{u} + \frac{3}{2} \frac{U_K^3 \lambda(u, x_4)}{u(u^3 - U_K^3)} \\
 &\quad - \frac{1}{3} \int \frac{Q_c g_s (\partial_{x_4} \lambda(u, x_4))}{u^3 \left(1 - \frac{U_K^3}{u^3}\right)^2} du \\
 G_1 &\rightarrow G_1 - \frac{3}{4} \frac{\lambda(u, x_4)}{u} - \frac{3}{2} \frac{U_K^3 \lambda(u, x_4)}{u(u^3 - U_K^3)} + \partial_u \lambda(u, x_4) \\
 C_1 &\rightarrow C_1 + \frac{1}{4} \frac{\lambda(u, x_4)}{u}
 \end{aligned} \tag{3.14}$$

leaves all equations of motion unchanged, and is exactly a coordinate change in  $u$  and  $x_4$ , namely

$$\begin{aligned}
 u &\rightarrow u + \lambda(u, x_4) \\
 x_4 &\rightarrow x_4 - \frac{1}{3} \int \frac{Q_c g_s (\partial_{x_4} \lambda(u, x_4))}{u^3 \left(1 - \frac{U_K^3}{u^3}\right)^2} du
 \end{aligned} \tag{3.15}$$

Hence, one of the degrees of freedom above is pure gauge. However, there is an added complication. If we eliminate  $\hat{\phi}_1$  using such a gauge transformation, the cusp in  $\hat{\phi}_1$  generates a delta function in the gauge transformation for  $B_1$ , and hence  $B_1$  is no longer a smooth function: it contains a delta function. Hence, we conclude that for the unsourced equations one may choose which degree of freedom to eliminate, but in the sourced equations, only  $B_1$  may be eliminated. However, as shown in appendix B, it is more convenient to not eliminate  $B_1$  completely, but rather to take  $B_1 = G_1$  as the choice.

Given the above transformations, we can immediately see that  $F_1$  and  $F_2$  of the last subsection are gauge independent. The remaining gauge dependent quantities  $\hat{\phi}_1$ ,  $B_1$  and  $G_1$  do not admit a gauge independent combination. Further, given the equations (3.13), only the equation  $B_1 = G_1$  is not gauge covariant. Therefore, we will sometimes refer to this as a gauge fixing.

### 4. Solutions: the linearized backreaction

Here we will analyze the differential equations of the last section in three separate cases:

1.  $U_K = 0$  decompactification limit: In this case we take  $U_K = 0$ , in a limiting sense of the background. In this limit  $R_x^2 \propto R_{(D4)}^3 / U_K$  becomes infinite, and so  $x_4$  decompactifies.



2.  $U_K = 0$ ,  $x_4$  compactified: In this case, we note that while the  $U_K = 0$  limit has decompactified  $x_4$ , one still has the isometry  $x_4 \rightarrow x_4 + \text{constant}$ . Hence, one may orbifold by this isometry and compactify  $x_4$ . We will parameterize this compactification using the same radius,  $R_x^2 = (4/9)R_{(D4)}^3/U_K$ . One way to think of this parametrization is that we have taken the spacetime to be that of  $U_K = 0$  while still requiring that  $x_4$  is compactified: we choose to parameterize the compactification of  $x_4$  such that we may compare easily to the  $U_K \neq 0$  case. In this way, we have taken the spacetime to be the cylinder to which the cigar asymptotes, and so this analysis gives the  $u \gg U_K$  behavior of the  $U_K \neq 0$  case. In all compact  $x_4$  cases, we will be considering the antipodal embedding,  $L = \pi R_x$ , which for concreteness we parameterize by the embedding  $x_4 = 0, \pi R_x$ .
3.  $U_K \neq 0$ : In this case, we analyze the equations as is. We make some basic comments about the nature of the Fourier transformed equations, and note that the point  $u = U_K$  is a regular singular point, and hence a finite convergent series about this point exists. We expand the solution about the tip of the cigar.

#### 4.1 $U_K = 0$ decompactification limit

In the  $U_K \rightarrow 0$  limit, the differential equations become

$$\begin{aligned}
 3\partial_u^2 F_1 + \frac{12}{u}\partial_u F_1 - \frac{54}{u^2}F_1 + \frac{Q_c g_s}{u^3}\partial_{x_4}^2 F_1 + \frac{2(3Q_c g_s)^{\frac{1}{2}}\Delta}{u^{\frac{3}{2}}} &= 0 \\
 3\partial_u^2 F_2 + \frac{12}{u}\partial_u F_2 + \frac{Q_c g_s}{u^3}\partial_{x_4}^2 F_2 - \frac{4(3Q_c g_s)^{\frac{1}{2}}\Delta}{u^{\frac{3}{2}}} &= 0 \\
 -4\partial_u^2 \hat{\phi}_1 - \frac{2}{u}\partial_u \hat{\phi}_1 - \frac{4Q_c g_s}{3u^3}\partial_{x_4}^2 \hat{\phi}_1 + \frac{4(3Q_c g_s)^{\frac{1}{2}}\Delta}{3u^{\frac{3}{2}}} &= 0.
 \end{aligned} \tag{4.1}$$

In the above equations, we take all functions to be functions of the form

$$\begin{aligned}
 F_i(u, x_4) &= uK_i(q) \quad q = \frac{x_4\sqrt{u}}{(Q_c g_s)^{\frac{1}{2}}} \\
 \hat{\phi}_1(u, x_4) &= uK_3(q).
 \end{aligned} \tag{4.2}$$

This has the effect of changing the delta function in  $x_4$  into a delta function in  $q$  as  $\Delta(x_4) = \Delta'(q)\sqrt{u/(Q_c g_s)}$ . Further, we take just a single brane so that  $\Delta' = \delta(q)$ . Taking the resulting equations, and multiplying them by  $u$ , we obtain ODE's with delta function sources

$$\begin{aligned}
 \left(\frac{3}{4}q^2 + 1\right)\partial_q^2 K_1 + \frac{33}{4}q\partial_q K_1 - 42K_1 + 2\sqrt{3}\delta(q) &= 0 \\
 \left(\frac{3}{4}q^2 + 1\right)\partial_q^2 K_2 + \frac{33}{4}q\partial_q K_2 + 12K_2 - 4\sqrt{3}\delta(q) &= 0 \\
 \left(\frac{3}{4}q^2 + 1\right)\partial_q^2 K_3 + 3q\partial_q K_3 + \frac{3}{2}K_3 - \sqrt{3}\delta(q) &= 0
 \end{aligned} \tag{4.3}$$

One constructs the delta function solution from the vacuum solution. The vacuum solutions to these equations may be written

$$\begin{aligned}
 K_1 &= K_a \left( \frac{1}{42} + \frac{1}{2}q^2 + q^4 \right) \\
 &\quad + K_b \frac{(3402q^{13} + 22113q^{11} + 57915q^9 + 77220q^7 + 54054q^5 + 18018q^3 + 2002q)}{(3q^2 + 4)^{\frac{9}{2}}} \\
 K_2 &= K_c \frac{\frac{280}{3}q \ln \left( \sqrt{3q^2 + 4} + q\sqrt{3} \right) + \sqrt{3q^2 + 4}\sqrt{3} \left( q^6 + \frac{19}{3}q^4 + \frac{58}{3}q^2 - \frac{128}{9} \right)}{(3q^2 + 4)^{\frac{9}{2}}} \\
 &\quad + K_d \frac{\frac{280}{3}q \ln \left( \sqrt{3q^2 + 4} - q\sqrt{3} \right) - \sqrt{3q^2 + 4}\sqrt{3} \left( q^6 + \frac{19}{3}q^4 + \frac{58}{3}q^2 - \frac{128}{9} \right)}{(3q^2 + 4)^{\frac{9}{2}}} \\
 K_3 &= K_e \frac{q}{3q^2 + 4} + K_f \frac{1}{3q^2 + 4}
 \end{aligned} \tag{4.4}$$

To obtain even (in  $q \rightarrow -q$ ) convergent (for large  $q$ ) quantities with cusps, we may construct the combinations

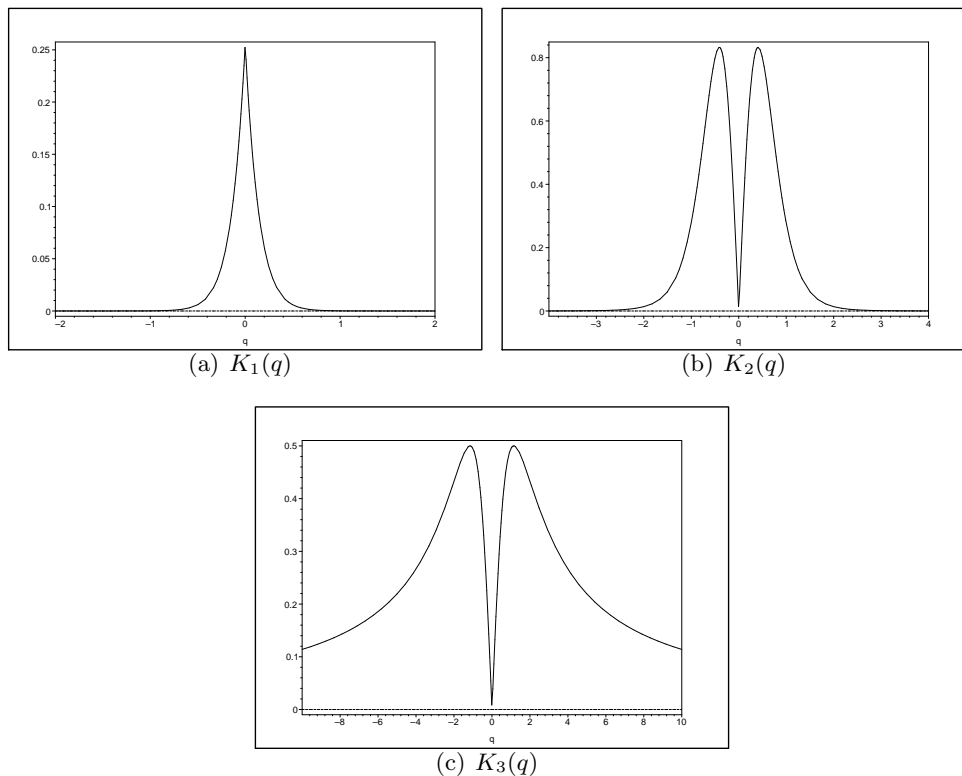
$$\begin{aligned}
 K_1 &= \frac{-256\sqrt{3}}{1001} \left( -\frac{3402}{3^{\frac{9}{2}}} \left( \frac{1}{42} + \frac{1}{2}q^2 + q^4 \right) \right. \\
 &\quad \left. + \left| \frac{(3402q^{13} + 22113q^{11} + 57915q^9 + 77220q^7 + 54054q^5 + 18018q^3 + 2002q)}{(3q^2 + 4)^{\frac{9}{2}}} \right| \right) \\
 K_2 &= 2^{10}\sqrt{3} \frac{|q|}{(3q^2 + 4)^{\frac{9}{2}}} \\
 &\quad + N_2 \frac{\frac{280}{3}q \ln \left( \frac{\sqrt{3q^2+4}+q\sqrt{3}}{\sqrt{3q^2+4}-q\sqrt{3}} \right) + 2\sqrt{3q^2+4}\sqrt{3} \left( q^6 + \frac{19}{3}q^4 + \frac{58}{3}q^2 - \frac{128}{9} \right)}{(3q^2 + 4)^{\frac{9}{2}}} \\
 K_3 &= 2\sqrt{3} \frac{|q|}{3q^2 + 4} + N_3 \frac{1}{3q^2 + 4}.
 \end{aligned} \tag{4.5}$$

The above have been written with the cusp solution first, and then an even function that converges (with coefficients  $N_i$ ). We have not been able to determine physical boundary conditions that fix  $N_i$ , and so we will leave them arbitrary when possible.

To graph them, however, we take  $N_i = 0$  and show these in figure 3. Further, note that the function  $K_3$  has a larger characteristic width, as it only converges as  $1/q$ . This will be important when we compactify  $x_4$ .

The height of the above functions grows as  $u$  because the peak happens at a fixed value of  $q$ , giving just a constant contribution times the dressing factor of  $u$ . Thus, one expects the perturbative approach to be valid up to  $u \ll 1/Q_f$ . This will be generic for later sections as well, as the decompactified behavior emerges at large  $u$  in the following sections.

We also wish to characterize the width of the ‘‘spike’’ in each graph. One way is to make sure that variations happen on scales larger than string scale. The slope of the graphs is largest in the vicinity of the spike, and this slope is determined by its  $q = 0$



**Figure 3:** Graphs of  $K_i(q)$ , setting  $N_2 = N_3 = 0$ .

behavior, and is therefore just a constant. Therefore, the  $x_4$  slope is simply  $d\phi_i/dx_4 = Q_f u^{3/2} R_{D4}^{-3/2} \times \text{constant}$ . The physical length that this corresponds to, however, is  $ds = (u/R_{D4})^{(3/4)} dx_4$ , and we require that  $d\phi_i/ds \ll 1/\ell_s$ . This gives  $u^3 \ll R_{D4}^3 / (Q_f^4 \ell_s^4) \propto R_{D4}^3 / (g_s^4 N_f^4)$ . Recalling the conditions above  $u \ll 1/Q_f$  and  $N_c \gg N_f$ , this condition follows, and so is not a new piece of information.

One may also wish to characterize the width when the linear part is no longer the dominant, and so characterize the width of when other “features” become important. This occurs when the  $q$  coordinate becomes order 1, and so translates into  $x_4 \propto R_{D4}^{3/2} / u^{1/2}$ . Again, translating this into a physical distance we find  $s = R_{D4}^{3/4} u^{1/4} \gg \ell_s$ , where we have required that this distance be greater than string scale. This gives a lower bound on  $u$ , however, it is the same lower bound coming from the Ricci scalar  $R \propto 1/(uR^3)^{1/2} \ll 1/\ell_s^2$  for the supergravity approximation. We see that we trust the supergravity to describe the backreaction above  $u \gg \ell_s^4 / R_{D4}^3$ , and that the perturbative results are good up to  $u \ll 1/Q_f$ .

Of course one may take the last two constraints on  $u$  and turn them into a unitless constraint on the parameters. We find that this is  $1/Q_f \gg \ell_s^4 / R_{D4}^3 \rightarrow g_s^2 N_f / N_c \ll 1$  which we can see is weaker than the other constraints  $N_f / N_c \ll 1$ ,  $g_s \ll 1$ .

#### 4.2 $U_K = 0$ with $x_4$ compactified

Here we will examine the  $U_K = 0$  case with  $x_4$  compactified as explained at the beginning of section 4. However, a few brief words are in order. We will do this case in two ways: by

summing the images from the last section, and by Fourier decomposing them. These two approaches are complimentary because one expects the sum on images to converge quickly for large  $u$  (when the images are well separated), and as we will see, the Fourier modes converge extremely quickly as  $u \rightarrow 0$ . Further, we will be considering the antipodal embedding,  $L = \pi R_x$ , which for concreteness we parameterize by the embedding  $x_4 = 0, \pi R_x$ .

**Sum images of decompactification.** Looking at the Fourier transform (below), there is no problem with compactifying  $x_4$ , however the solution for  $K_3$  above seems to have a problem. Note that if we take the above  $K_3$  and sum over images in  $q$  for some fixed value of  $u$ , we expect a divergence, as the function only converges as  $1/q$ . However, switching back to the  $u, x_4$  language, we find that the  $n^{\text{th}}$  image of  $K_3$  is

$$\hat{\phi}_{n,1} = \frac{3 \frac{u^{\frac{3}{2}}}{U_K^{\frac{1}{2}}} \left| \frac{x_4}{R_x} + \pi n \right|}{\frac{u}{U_K} \left( \frac{x_4}{R_x} + \pi n \right)^2 + 9} \rightarrow \frac{3\sqrt{u}\sqrt{U_K}}{\pi |n|} \quad (4.6)$$

where the right hand side is it the large  $n$  behavior. This, however, is actually a homogeneous solution to the original differential equation we started with. This suggests a solution to this difficulty, and we take instead

$$\hat{\phi}_{n,1} = \frac{3 \frac{u^{\frac{3}{2}}}{U_K^{\frac{1}{2}}} \left| \frac{x_4}{R_x} + \pi n \right|}{\frac{u}{U_K} \left( \frac{x_4}{R_x} + \pi n \right)^2 + 9} - \frac{3\sqrt{u}\sqrt{U_K}}{\pi |n + C_n|} \quad (4.7)$$

where this is just adding a zero mode of the differential equation. The sum of this function in  $n$  converges, as the large  $n$  behavior is order  $1/n^2$ .

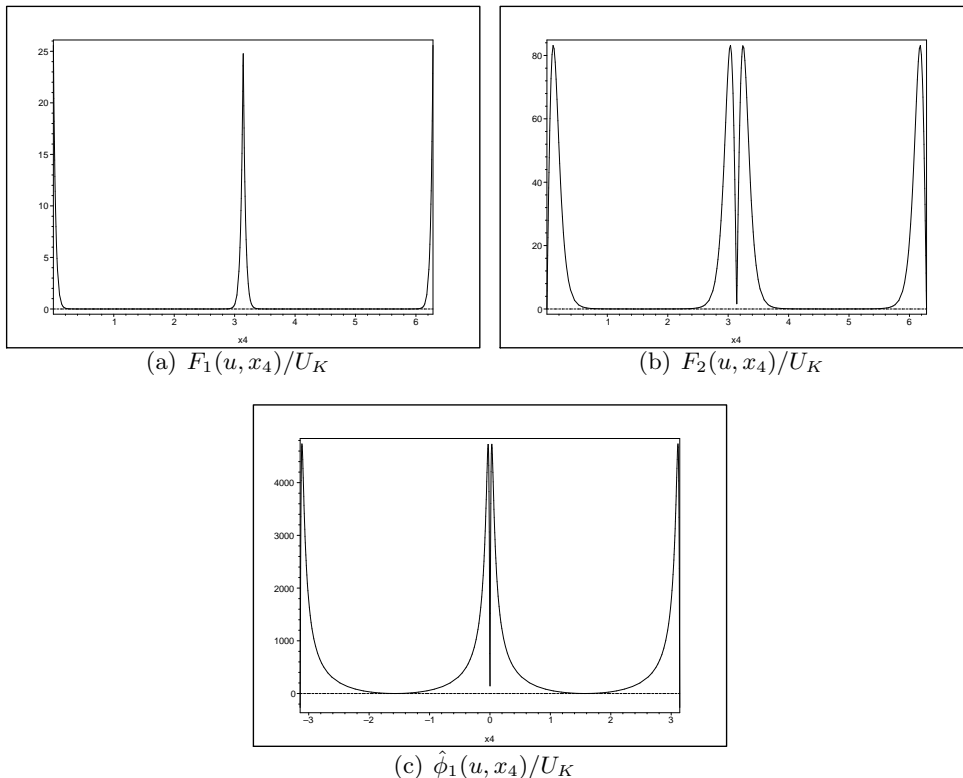
Now we make one more final comment. To fix  $C_n$ , we require that  $\lim_{u \rightarrow \infty} \hat{\phi}_1(u, x_4 = \pi R_x/2) = 0$ . This is roughly requiring that when you are as far away from the branes as possible that the perturbation should be 0. The solution to this constraint is  $C_n = 1/2$ . Of course a different set of  $C_n$  could be chosen in such a way as to not affect the sum: this, by definition, is unphysical, as none of the field values would change.

Therefore, for the compactified case, we take

$$\hat{\phi}_1(u, x_4) = \sum_{n=-\infty}^{\infty} \left( \frac{3 \frac{u^{\frac{3}{2}}}{U_K^{\frac{1}{2}}} \left| \frac{x_4}{R_x} + \pi n \right|}{\frac{u}{U_K} \left( \frac{x_4}{R_x} + \pi n \right)^2 + 9} - \frac{3\sqrt{u}\sqrt{U_K}}{\pi |n + \frac{1}{2}|} \right). \quad (4.8)$$

The other functions we take  $N_2 = 0$  and simply sum on images. We plot these in figure 4.

In figure 4 we have summed many fewer modes for  $F_1$  because this function converges very fast (as  $1/q^{14}$ ), and so fewer images are needed. Further, we can see that one must go to much larger values of  $u$  to obtain well separated features for  $\hat{\phi}_1$ . One could have guessed this from the results of the decompactified case: the characteristic width of the functions  $K_3(q)$  (associated with  $\hat{\phi}_1$ ) is much larger than for the other functions  $K_1$  and  $K_2$ . However, one can see the shape of the functions approaching the decompactified case:



**Figure 4:**  $F_1/U_K$  as a function of  $x_4/R_x$  for  $u/U_K = 100$  using images from  $n = -50 \dots 50$ ,  $F_2/U_K$  as a function of  $x_4/R_x$  for  $u/U_K = 100$ ,  $n = -500 \dots 500$  and  $\hat{\phi}_1/U_K$  as a function of  $x_4/R_x$  for  $u/U_K = 10,000$ ,  $n = -500 \dots 500$ .

$F_1$  has a single cusp like  $-|x_4|$  and  $\hat{\phi}_1$  and  $F_2$  have cusps like  $|x_4|$  surrounded by two maxima. We show plots for various  $u$  in the next subsection where we consider the Fourier transform of the equations.

**Fourier decomposition.** We start with the same equations as the last subsection

$$3\partial_u^2 F_1 + \frac{12\partial_u F_1}{u} + \frac{g_s Q_c \partial_{x_4}^2 F_1}{u^3} - \frac{54F_1}{u^2} + \frac{2(Q_c g_s)^{\frac{2}{3}} \sqrt{3}}{u^2} \sqrt{\frac{u}{(Q_c g_s)^{\frac{1}{3}}}} \Delta = 0 \quad (4.9)$$

$$3\partial_u^2 F_2 + \frac{12\partial_u F_2}{u} + \frac{g_s Q_c \partial_{x_4}^2 F_2}{u^3} - \frac{4(Q_c g_s)^{\frac{2}{3}} \sqrt{3}}{u^2} \sqrt{\frac{u}{(Q_c g_s)^{\frac{1}{3}}}} \Delta = 0 \quad (4.10)$$

$$-4\partial_u^2 \hat{\phi}_1 - \frac{4}{3} \frac{Q_c g_s \partial_{x_4}^2 \hat{\phi}_1}{u^3} - \frac{2\partial_u \hat{\phi}_1}{u} + \frac{\frac{4}{3}(Q_c g_s)^{\frac{2}{3}} \sqrt{3}}{u^2} \sqrt{\frac{u}{(Q_c g_s)^{\frac{1}{3}}}} \Delta = 0 \quad (4.11)$$

and Fourier transform them

$$3\partial_u^2 F_{m,1} + \frac{12\partial_u F_{m,1}}{u} - \frac{m^2 \frac{27U_K}{4} F_{m,1}}{u^3} - \frac{54F_{m,1}}{u^2} + \frac{9\sqrt{U_K}(1+(-1)^m)}{2\pi u^{\frac{3}{2}}} = 0 \quad (4.12)$$

$$3\partial_u^2 F_{m,2} + \frac{12\partial_u F_{m,2}}{u} - \frac{m^2 \frac{27U_K}{4} F_{m,2}}{u^3} - \frac{18\sqrt{U_K}(1+(-1)^m)}{2\pi u^{\frac{3}{2}}} = 0 \quad (4.13)$$

$$-4\partial_u^2 \hat{\phi}_{m,1} + \frac{4m^2 \frac{27U_K}{4} \hat{\phi}_{m,1}}{3u^3} - \frac{2\partial_u \hat{\phi}_{m,1}}{u} + \frac{6\sqrt{U_K}(1+(-1)^m)}{2\pi u^{\frac{3}{2}}} = 0. \quad (4.14)$$

where we have assumed that  $x_4$  is periodic with

$$x_4 = x_4 + 2\pi R_x, \quad R_x^2 = \frac{4}{27} \frac{Qcgs}{U_K} = \frac{4}{9} \frac{R_{D4}^3}{U_K} \quad (4.15)$$

as in the last section (the reason for this parameterization is explained at the beginning of section 4).

The above equations can be brought to a simple and familiar form by the following change of coordinates and functions

$$\begin{aligned} F_{m,1}(u) &= \frac{G_{m,1} \left( \sqrt{\frac{9m^2 U_K}{u}} \right)}{u^{\frac{3}{2}}} \\ F_{m,2}(u) &= \frac{G_{m,2} \left( \sqrt{\frac{9m^2 U_K}{u}} \right)}{u^{\frac{3}{2}}} \\ \hat{\phi}_{m,1}(u) &= G_{m,3} \left( \sqrt{\frac{9m^2 U_K}{u}} \right) u^{\frac{1}{2}} \\ u &= \frac{9m^2 U_K}{\rho^2} \end{aligned}$$

The above differential equations become

$$\rho^2 \partial_\rho^2 G_{m,1} + \rho \partial_\rho G_{m,1} + (-\rho^2 - 9^2) G_{m,1} = -\frac{972(1+(-1)^m)m^4 U_K^{\frac{5}{2}}}{2\pi \rho^4} \quad (4.16)$$

$$\rho^2 \partial_\rho^2 G_{m,2} + \rho \partial_\rho G_{m,2} + (-\rho^2 - 3^2) G_{m,1} = \frac{1944(1+(-1)^m)m^4 U_K^{\frac{5}{2}}}{2\pi \rho^4} \quad (4.17)$$

$$\partial_\rho^2 G_{m,3} - G_{m,3} = \frac{6(1+(-1)^m)U_K^{\frac{1}{2}}}{2\pi \rho^2}. \quad (4.18)$$

These then are of two forms: modified Bessel equations with a simple monomial source,<sup>9</sup> and an exponential function with a simple monomial source. The general solution to the

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<sup>9</sup>Solutions to these equations are known as (modified) Lommel functions. However they are related to generalized hypergeometric functions. We opt to use the notation of hypergeometric series, as these are more general, and perhaps more familiar to the reader.

above equations is [42]

$$G_{m,1} = C_1 I_9(\rho) + C_2 K_9(\rho) - \frac{972(1 + (-1)^m)m^4 U_K^{\frac{5}{2}}}{2\pi\rho^4(-13)(5)} {}_1F_2\left(1; -\frac{11}{2}, \frac{7}{2}; \frac{\rho^2}{4}\right) \quad (4.19)$$

$$G_{m,2} = C_3 I_3(\rho) + C_4 K_3(\rho) + \frac{1944(1 + (-1)^m)m^4 U_K^{\frac{5}{2}}}{2\pi\rho^4(-7)(-1)} {}_1F_2\left(1; -\frac{5}{2}, \frac{1}{2}; \frac{\rho^2}{4}\right) \quad (4.20)$$

$$G_{m,3} = C_5 e^\rho + C_6 e^{-\rho} + \frac{3(1 + (-1)^m)(e^\rho \text{Ei}(1, \rho) + e^{-\rho} \text{Re}(\text{Ei}(1, -\rho))) U_K^{\frac{1}{2}}}{2\pi} \quad (4.21)$$

where  $I$  and  $K$  are the modified Bessel functions,  ${}_1F_2$  is the generalized hypergeometric function,<sup>10</sup> and  $\text{Ei}(a, x)$  is the exponential integral function.

$$\text{Ei}(a, x) = \int_1^\infty e^{-yx} y^{-a} dy \quad (4.22)$$

and we have used  $\text{Re}(\text{Ei}(1, -\rho)) = \text{Ei}(1, -\rho) + \pi i$  to remove the  $-\pi i$  associated with going around the branch point at  $x = 0$ .

Here, one may worry about the convergence of the above Fourier decomposition because the coefficient above depend on  $m$  in positive powers. However, recall that we wish to sum on  $m$  for fixed  $u$ , not fixed  $\rho$ . In fact the factor of  $m$  completely cancels out of the above coefficients once returning to the  $u$  coordinate ( $\rho^4 \propto m^4/u^2$ ). The only  $m$  dependence comes about in the arguments of the homogeneous and non homogeneous terms. Therefore, we may effectively analyze convergence of the Fourier modes as convergence in the variable  $\rho \rightarrow \infty$ , as this is the limit to which  $m \rightarrow \infty$  corresponds. The inhomogeneous solution to (4.21) is indeed convergent and admits a power series expansion about  $\rho = \infty$ . Therefore one does not wish to turn on the growing exponential (this would not converge summing on  $m$ ), and the shrinking exponential is simply negligible. Hence, we may set  $C_5 = C_6 = 0$  for all  $m$  and get a convergent series.

The remaining equations, however, deserve some special treatment. The bessel equation (and the equation for  ${}_1F_2$ ) have an essential singularity at  $\rho = \infty$ . Therefore we consider the asymptotics of the above functions for large  $\rho$ :

$$\begin{aligned} I_\nu(\rho) &\rightarrow \frac{1}{\sqrt{2\pi\rho}} e^\rho \\ K_\nu(\rho) &\rightarrow \frac{1}{\sqrt{2\pi\rho}} e^{-\rho} \end{aligned} \quad (4.23)$$

and [43]

$${}_1F_2(a_1; b_1, b_2; x) \rightarrow \frac{\Gamma(b_1)\Gamma(b_2)}{\sqrt{\pi}\Gamma(a_1)} (-x)^{\frac{1}{2}(a_1-b_1-b_2)} \cos\left(\frac{\pi}{2}\left(a_1 - b_1 - b_2 + \frac{1}{2}\right) + 2\sqrt{-x}\right). \quad (4.24)$$

---

<sup>10</sup>we do in fact mean  ${}_1F_2$ , not  ${}_2F_1$

After substituting in  $x = \rho^2/2$  and choosing appropriate branches for the square roots  $((-1)^{1/2} = +i)$ , one finds

$$\begin{aligned} \frac{{}_1F_2(1; -\frac{11}{2}, \frac{7}{2}; \frac{\rho^2}{4})}{\rho^4} &\rightarrow \frac{\sqrt{\pi}\sqrt{2}}{2772\rho^{\frac{1}{2}}}e^\rho \\ \frac{{}_1F_2(1; -\frac{5}{2}, \frac{1}{2}; \frac{\rho^2}{4})}{\rho^4} &\rightarrow -\frac{\sqrt{\pi}\sqrt{2}}{60\rho^{\frac{1}{2}}}e^\rho. \end{aligned} \quad (4.25)$$

We therefore use the following combinations

$$G_{m,1} = C_2 K_9(\rho) - \frac{972(1 + (-1)^m)m^4 U_K^{\frac{5}{2}}}{2\pi(-13)(5)} \left( \frac{{}_1F_2\left(1; -\frac{11}{2}, \frac{7}{2}; \frac{\rho^2}{4}\right)}{\rho^4} - \frac{\pi I_9(\rho)}{1386} \right) \quad (4.26)$$

$$G_{m,2} = C_4 K_3(\rho) + \frac{1944(1 + (-1)^m)m^4 U_K^{\frac{5}{2}}}{2\pi(-7)(-1)} \left( \frac{{}_1F_2\left(1; -\frac{5}{2}, \frac{1}{2}; \frac{\rho^2}{4}\right)}{\rho^4} + \frac{\pi I_3(\rho)}{30} \right) \quad (4.27)$$

$$G_{m,3} = C_6 e^{-\rho} + \frac{3(1 + (-1)^m)(e^\rho \text{Ei}(1, \rho) + e^{-\rho} \text{Re}(\text{Ei}(1, -\rho)))U_K^{\frac{1}{2}}}{2\pi}. \quad (4.28)$$

It is now a simple matter to replace the definition of  $\rho$  above and sum the series. Although we offer no analytic proof here, the above functions can be seen to converge quickly enough for large  $m$ . We note that because the above functions are functions of  $m^2/u$ , the convergence in  $m$  and  $u \rightarrow 0$  are connected. As promised, the Fourier expansion converges more quickly for smaller  $u$ .

In the last two equations, it should be noted that for large  $u$ , the particular solution selected dominates over the remaining inhomogeneous solution. However, in the first equation, the homogeneous solution dominates. In the following, we still set  $C_2 = C_4 = C_6 = 0$ .

Solving the  $m = 0$  case is trivial, but for completeness, we give the solutions for this as well

$$\begin{aligned} F_{0,1} &= -\frac{12u^{\frac{1}{2}}U_K^{\frac{1}{2}}}{65\pi} + C_{0,5}u^3 + \frac{C_{0,6}}{u^6} \\ F_{0,2} &= -\frac{24u^{\frac{1}{2}}U_K^{\frac{1}{2}}}{7\pi} + \frac{C_{0,3}}{u^3} + C_{0,4} \end{aligned} \quad (4.29)$$

$$\hat{\phi}_{0,1} = -\frac{3U_K^{\frac{1}{2}}\ln(u)u^{\frac{1}{2}}}{\pi} + \frac{6u^{\frac{1}{2}}U_K^{\frac{1}{2}}}{\pi} + 2C_{0,1}u^{\frac{1}{2}} + C_{0,2} \quad (4.30)$$

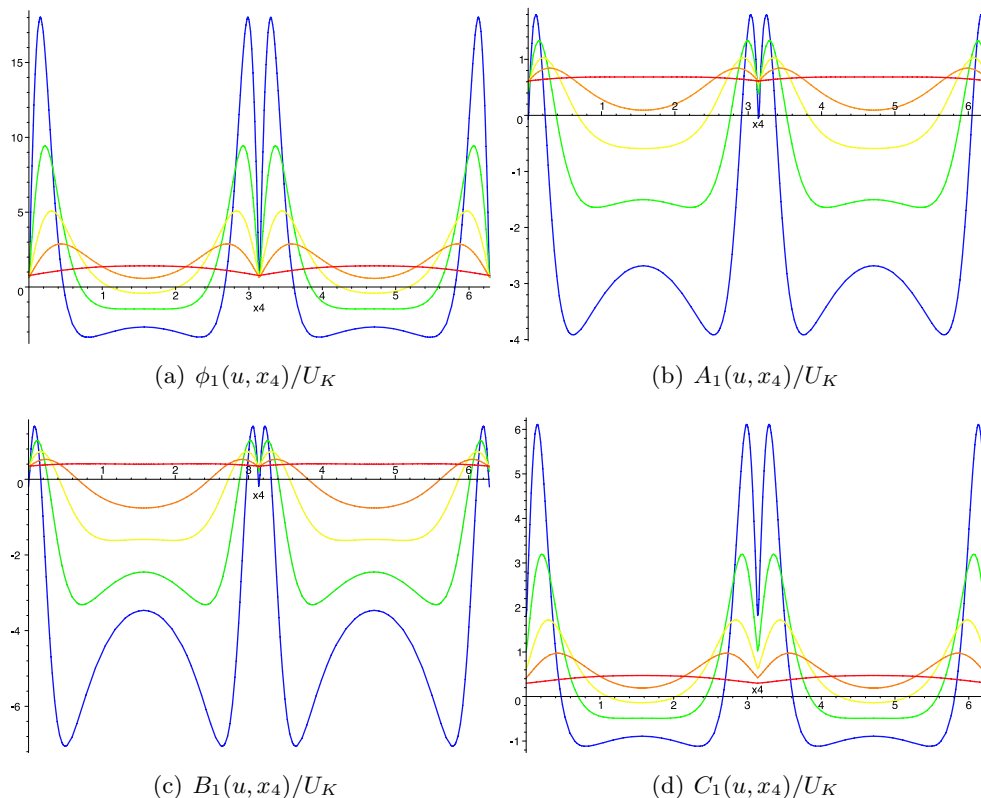
and we again set the unfixed constants above to 0.

We show here the plots of the physical fields  $\phi_1, A_1, B_1 = G_1, C_1$  for the first 150 modes in figure 5.

We may wish to ask what the long distance behavior is in  $u/U_K$ . For this, we plot the long distance behavior of the independent modes  $F_1, F_2, \hat{\phi}_1$  in figure 6.

The graphs in figure 6 are directly comparable to the plots in figure 4, however there may be discrepancies between the two. For example, in figure 6 one notes that the function  $\hat{\phi}$  does not quite touch 0 at  $x_4 = \pi R_x/2$  as it is required to for figure 4. This is merely a





**Figure 5:** Graphs of  $\phi_1/U_K$ ,  $A_1/U_K$ ,  $B_1/U_K = G_1/U_K$ ,  $C_1/U_K$  graphed as a function of  $x_4/R_x$ . Each graph represents a different  $u$  value of  $u = 1, 5, 10, 20, 40$  in ROYGB order. All graphs use the first 150 modes ( $m = 0 \dots 150$ ), of which only even  $m$  are non zero.

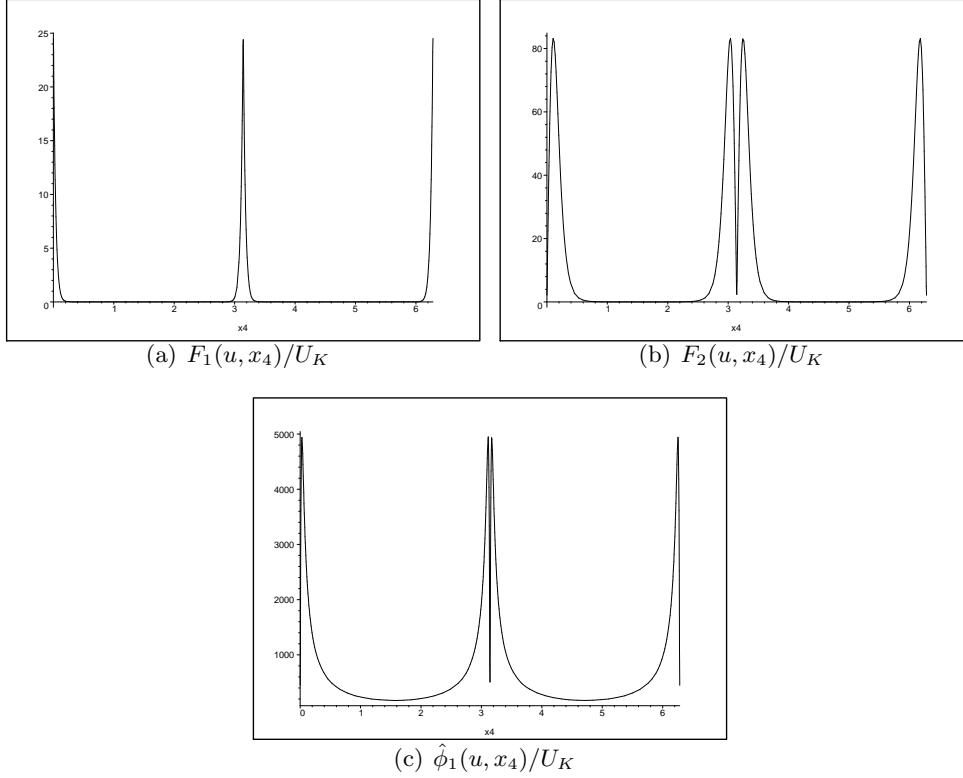
matter of setting the correct zero mode for the homogenous equation in this case. However, to make the plots match *exactly* in all cases, one may, if one wishes, Poisson resum the results from the last subsection, which will fix all constants above in terms of the constants from the last subsection. We, however, have not done this.

We should note also that the above analysis allows for more generality than summing the images. In the decompactification case a very specific functional form was taken, to which many homogenous solutions do not conform. This property is inherited when summing on images. By the orthogonality and completeness of the trigonometric functions, we are guaranteed to generate all solutions to the original gravitational ansatz when using the Fourier analysis.

### 4.3 $U_K \neq 0$

**Fourier decomposition.** Fourier decomposing the general equations at the beginning of section 3 gives

$$3\partial_u^2 F_{m,1} + \frac{3(4u^3 - U_K^3)\partial_u F_{m,1}}{u(u^3 - U_K^3)} - \frac{m^2 \frac{27U_K}{4} u^3 F_{m,1}}{(u^3 - U_K^3)^2}$$



**Figure 6:**  $F_1/U_K$  as a function of  $x_4/R_x$  for  $u/U_K = 100$ ,  $F_2/U_K$  as a function of  $x_4/R_x$  for  $u/U_K = 100$  and  $\hat{\phi}_1/U_K$  as a function of  $x_4/R_x$  for  $u/U_K = 10,000$  (bottom) all graphed using the first 400 modes ( $m = 0 \dots 400$ ).

$$-\frac{54uF_{m,1}}{(u^3 - U_K^3)} + \frac{9u\sqrt{U_K}(1 + (-1)^m)}{2\pi(u^3 - U_K^3)} \sqrt{\frac{u}{\left(1 - \frac{U_K^3}{u^3}\right)}} = 0 \quad (4.31)$$

$$3\partial_u^2 F_{m,2} + \frac{3(4u^3 - U_K^3)\partial_u F_{m,2}}{u(u^3 - U_K^3)} - \frac{m^2 \frac{27U_K}{4} u^3 F_2}{(u^3 - U_K^3)^2} - \frac{18u\sqrt{U_K}(1 + (-1)^m)}{2\pi(u^3 - U_K^3)} \sqrt{\frac{u}{\left(1 - \frac{U_K^3}{u^3}\right)}} = 0 \quad (4.32)$$

$$-4\partial_u^2 \hat{\phi}_{m,1} - \frac{2(u^3 - 7U_K^3)\partial_u \hat{\phi}_{m,1}}{u(u^3 - U_K^3)} - \frac{36uU_K^3 \hat{\phi}_{m,1}}{(u^3 - U_K^3)^2} + \frac{4m^2 u^3 \frac{27U_K}{4} \hat{\phi}_{m,1}}{3(u^3 - U_K^3)^2} + \frac{6u\sqrt{U_K}(1 + (-1)^m)}{2\pi(u^3 - U_K^3)} \sqrt{\frac{u}{\left(1 - \frac{U_K^3}{u^3}\right)}} = 0. \quad (4.33)$$

where now the periodicity of  $x_4$

$$x_4 = x_4 + 2\pi R_x, \quad R_x^2 = \frac{4}{27} \frac{Q_c g_s}{U_K} = \frac{4}{9} \frac{R_{D4}^3}{U_K} \quad (4.34)$$

is dictated by smoothness of the gravitational solution near the tip of the cigar ( $u \rightarrow U_K$ ).

Recall that we will be considering the antipodal embedding  $L = \pi R_x$ , so that the delta functions are located at  $x_4 = 0, \pi R_x$ .

It is clear from the above that these functions are functions only of  $u/U_K$  as  $U_K$  is the only dimensionful parameter left. Under the coordinate change  $\hat{u} = \frac{u}{U_K}$  and the redefinition of fields  $F_i = \hat{F}_i \times U_K$  and  $\hat{\phi}_1 = \tilde{\phi}_1 \times U_K$  we find

$$3\partial_{\hat{u}}^2 \hat{F}_{m,1} + \frac{3(4\hat{u}^3 - 1)\partial_{\hat{u}} \hat{F}_{m,1}}{\hat{u}(\hat{u}^3 - 1)} - \frac{27m^2 \hat{u}^3 \hat{F}_{m,1}}{4(\hat{u}^3 - 1)^2} - \frac{54\hat{u} \hat{F}_{m,1}}{(\hat{u}^3 - 1)} + \frac{9\hat{u}^3(1 + (-1)^m)}{2\pi(\hat{u}^3 - 1)^{\frac{3}{2}}} = 0 \quad (4.35)$$

$$3\partial_{\hat{u}}^2 \hat{F}_{m,2} + \frac{3(4\hat{u}^3 - 1)\partial_{\hat{u}} \hat{F}_{m,2}}{\hat{u}(\hat{u}^3 - 1)} - \frac{27m^2 \hat{u}^3 \hat{F}_{m,2}}{4(\hat{u}^3 - 1)^2} - \frac{18\hat{u}^3(1 + (-1)^m)}{2\pi(\hat{u}^3 - 1)^{\frac{3}{2}}} = 0 \quad (4.36)$$

$$-4\partial_{\hat{u}}^2 \tilde{\phi}_{m,1} - \frac{2(\hat{u}^3 - 7)\partial_{\hat{u}} \tilde{\phi}_{m,1}}{\hat{u}(\hat{u}^3 - 1)} - \frac{36\hat{u} \tilde{\phi}_{m,1}}{(\hat{u}^3 - 1)^2} + 9\frac{m^2 \hat{u}^3 \tilde{\phi}_{m,1}}{(\hat{u}^3 - 1)^2} + \frac{6\hat{u}^3(1 + (-1)^m)}{2\pi(\hat{u}^3 - 1)^{\frac{3}{2}}} = 0. \quad (4.37)$$

The above differential equations are difficult to solve, even excluding the inhomogeneous piece. The homogeneous parts of the equations can be seen to have 5 regular singular points at  $\hat{u} = 0, \omega^0, \omega^1, \omega^2, \infty$  where  $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$  is a third root of unity. However, as all of the singularities are regular, one may go about finding a Laurent series expansion about any given point (using standard textbook techniques). Rather than doing this in the Fourier basis, we find that it is easier to leave the functions of  $x_4$  intact, and expand the function about a specific  $u$  value, and then solve for the functions of  $x_4$  multiplying each power individually.

**$u \rightarrow U_K$  expansion.** First, for expanding around the point  $u = U_K$ , and to make contact with much of the literature, we change to the coordinate  $z$  defined by

$$u^3 = U_K^3 + U_K z^2. \quad (4.38)$$

In these coordinates, the (non algebraic) equations read

$$\frac{Q_c g_s (U_K^2 + z^2) \partial_{x_4}^2 F_1}{(U_K^3 + U_K z^2)^{\frac{1}{3}} z^4 U_K} + \frac{27(U_K^2 + z^2) \partial_z^2 F_1}{4 U_K z^2} + \frac{27(U_K^2 + 3z^2) \partial_z F_1}{4 U_K z^3} + \frac{54 F_1}{U_K z^2} - \frac{2(3Q_c g_s)^{\frac{1}{2}} \sqrt{(U_K(U_K^2 + z^2)^4)^{\frac{1}{3}} \Delta}}{U_K z^3} = 0 \quad (4.39)$$

$$\frac{Q_c g_s (U_K^2 + z^2) \partial_{x_4}^2 F_2}{(U_K^3 + U_K z^2)^{\frac{1}{3}} z^4 U_K} + \frac{27(U_K^2 + z^2) \partial_z^2 F_2}{4 U_K z^2} + \frac{27(U_K^2 + 3z^2) \partial_z F_2}{4 U_K z^3} - \frac{4(3Q_c g_s)^{\frac{1}{2}} \sqrt{(U_K(U_K^2 + z^2)^4)^{\frac{1}{3}} \Delta}}{U_K z^3} = 0 \quad (4.40)$$

$$-\frac{4 Q_c g_s (U_K^2 + z^2) \partial_{x_4}^2 \hat{\phi}_1}{3 (U_K^3 + U_K z^2)^{\frac{1}{3}} z^4 U_K} - \frac{9 (U_K^2 + z^2) \partial_z^2 \hat{\phi}_1}{U_K z^2} - 3 \frac{(2z^2 - 9U_K^2) \partial_z \hat{\phi}_1}{U_K z^3} - \frac{36 U_K \hat{\phi}_1}{z^4} + \frac{\frac{4}{3} (3Q_c g_s)^{\frac{1}{2}} \sqrt{(U_K(U_K^2 + z^2)^4)^{\frac{1}{3}} \Delta}}{U_K z^3} = 0 \quad (4.41)$$

$$B_1 = G_1 = -\frac{1}{5} F_1 + \frac{1}{5} F_2 - \frac{3}{5} \frac{\partial_z \hat{\phi}_1 (U_K^2 + z^2)}{z} + \frac{1}{10} \frac{\hat{\phi}_1 (6U_K^2 - z^2)}{z^2} \quad (4.42)$$

after a bit of cleaning up.

Now, to expand about  $z = 0$ , we expand the functions multiplying each differential operator that acts on the fields  $F_i, \hat{\phi}$  above.

$$\frac{Q_c g_s \partial_{x_4}^2 F_1}{z^4} + \frac{27 U_K \partial_z^2 F_1}{4 z^2} + \frac{27 U_K \partial_z F_1}{4 z^3} - \frac{54 F_1}{U_K z^2} + \frac{2(3Q_c g_s)^{\frac{1}{2}} \sqrt{U_K} \Delta}{z^3} = 0 \quad (4.43)$$

$$\frac{Q_c g_s \partial_{x_4}^2 F_2}{z^4} + \frac{27 U_K \partial_z^2 F_2}{4 z^2} + \frac{27 U_K \partial_z F_2}{4 z^3} - \frac{4(3Q_c g_s)^{\frac{1}{2}} \sqrt{U_K} \Delta}{z^3} = 0 \quad (4.44)$$

$$-\frac{4}{3} \frac{Q_c g_s \partial_{x_4}^2 \hat{\phi}_1}{z^4} - \frac{9 U_K \partial_z^2 \hat{\phi}_1}{z^2} + \frac{27 U_K^2 \partial_z \hat{\phi}_1}{z^3} - \frac{36 U_K \hat{\phi}_1}{z^4} + \frac{\frac{4}{3} (3Q_c g_s)^{\frac{1}{2}} \sqrt{U_K} \Delta}{z^3} = 0 \quad (4.45)$$

$$B_1 = G_1 = -\frac{1}{5} F_1 + \frac{1}{5} F_2 - \frac{3 U_K^2 \partial_z \hat{\phi}_1}{5 z} + \frac{3 U_K^2 \hat{\phi}_1}{5 z^2} \quad (4.46)$$

We will now assume that all of the Fourier modes of the above fields have expansions about  $z = 0$  with a finite number of negative powers. This allows us to simply count powers in  $z$  and neglect any terms that are not of leading order. For example, this allows us to drop the term  $F_1/z^2$  above. The rest of the equation is homogeneous in  $z$  only if  $F_1 \sim z$  as  $z \rightarrow 0$ . Likewise, we conclude that to leading order  $F_i = z L_i(x_4)$  and  $\hat{\phi}_1 = z L_3(x_4)$ . To leading order in  $z$ , the above equations become

$$\partial_{x_4} \partial_{x_4} L_1(x_4) + \frac{27 U_K}{4 Q_c g_s} L_1(x_4) + 2 \sqrt{\frac{3 U_K}{Q_c g_s}} \Delta = 0 \quad (4.47)$$

$$\partial_{x_4} \partial_{x_4} L_2(x_4) + \frac{27 U_K}{4 Q_c g_s} L_2(x_4) - 4 \sqrt{\frac{3 U_K}{Q_c g_s}} \Delta = 0 \quad (4.48)$$

$$\partial_{x_4} \partial_{x_4} L_3(x_4) + \frac{27 U_K}{4 Q_c g_s} L_3(x_4) - \sqrt{\frac{3 U_K}{Q_c g_s}} \Delta = 0 \quad (4.49)$$

$$B_1 = G_1 = z \frac{1}{5} (L_2 - L_1). \quad (4.50)$$

These are easily solved.

$$L_1(x_4) = -\frac{4}{3} \left| \sin \left( \frac{x_4}{R_x} \right) \right| \quad (4.51)$$

$$L_2(x_4) = +\frac{8}{3} \left| \sin \left( \frac{x_4}{R_x} \right) \right| \quad (4.52)$$

$$L_3(x_4) = +\frac{2}{3} \left| \sin \left( \frac{x_4}{R_x} \right) \right| \quad (4.53)$$

$$R_x^2 = \frac{4 Q_c g_s}{27 U_K} \quad (4.54)$$

where  $R_x$  is the periodicity of  $x_4 \equiv x_4 + 2\pi R_x$  defined before. Note that this is exactly the kind of behavior one would expect because  $z |\sin(x_4/R_x)| = |y_1|$  when switching to “cartesian”  $y_2 = z \cos(x_4/R_x), y_1 = z \sin(x_4/R_x)$  coordinates, and is similar qualitatively to a D8 in flat space.

One may continue this process and in fact get the above functions to the next order. From the general differential equation in  $z$ , one may easily read that the expansion will be in terms of odd powers of  $z$ . This is because the functions being expanded are all  $(U_K^2 + z^2)^n$ , and so coefficients of even (odd) powers only mix with coefficients of other even (odd) powers. The next term, therefore, should be of order  $z^3$ . We solve the resulting equations, and find

$$\begin{aligned}
 F_1(z, x_4) &= -\frac{2}{3}z \sin\left(\frac{|x_4|}{R_x}\right) - \frac{17}{108} \frac{z^3 \left(3 \sin\left(\frac{|x_4|}{R_x}\right) - \sin\left(\frac{3|x_4|}{R_x}\right)\right)}{U_K^2} + \mathcal{O}(z^5) \\
 F_2(z, x_4) &= 2 \times \frac{2}{3}z \sin\left(\frac{|x_4|}{R_x}\right) - \frac{14}{108} \frac{z^3 \left(3 \sin\left(\frac{|x_4|}{R_x}\right) - \sin\left(\frac{3|x_4|}{R_x}\right)\right)}{U_K^2} + \mathcal{O}(z^5) \\
 \hat{\phi}_1(z, x_4) &= \frac{1}{3}z \sin\left(\frac{|x_4|}{R_x}\right) + \mathcal{O}(z^5).
 \end{aligned} \tag{4.55}$$

One may analyze the above functions for the various length scales of these cusps. One finds that they slope is directly limited by  $d\phi_i/ds = d\phi_i/dy_1 (U_K/R_{D4})^{3/4} \sim (U_K/R_{D4})^{3/4} Q_f \ll 1/\ell_s$ . This gives  $(g_5^4 T_{\text{st}})^{1/2} N_f \propto (T_{\text{st}}/M_{\text{gb}}^2)^{1/2} \lambda_4 N_f/N_c \ll 1$ .

Further, the onset of new “features” is at  $z^2 = U_K^2$ , which we require to be a large physical length:  $U_K^2 (R_{D4}/U_K)^{3/2} \gg \ell_s^2$ , which is equivalent to the original condition  $\lambda_5 \gg R_x$  given for the supergravity limit. Hence, we trust the supergravity approximation to describe the features near  $u \rightarrow U_K$ .

## 5. Analysis of equations: stability

A few words are in order to explain how we will address the issue of stability. We will not solve the eigenvalue problem, numerically or otherwise, to establish the four dimensional masses for fluctuations as being positive definite with a mass gap. Instead, we will simply show that to quadratic order in fluctuations, all actions are of the form (up to gauge)

$$\int d^5\xi \left[ -\mathcal{F}_0(\xi; Q_f) - \mathcal{F}_1(\xi; Q_f) \left( -(\partial_t M)^2 + \sum_{i=1}^3 (\partial_{x_i} M)^2 \right) - \mathcal{F}_2(\xi; Q_f) (\partial_{\xi^5} M)^2 - \mathcal{F}_3(\xi; Q_f) M^2 \right] \tag{5.1}$$

where  $M$  is a field describing the fluctuation, and

$$\mathcal{F}_i(\xi; Q_f = 0) > 0. \tag{5.2}$$

This last statement is important, because it implies that anywhere the perturbative analysis is valid that (to linear order in  $Q_f$ )

$$\mathcal{F}_i(\xi; Q_f) = \mathcal{F}_i(\xi; Q_f = 0) \left( 1 + \frac{Q_f}{\mathcal{F}_i} (\partial_{Q_f} \mathcal{F}_i) |_{Q_f=0} \right) > 0. \tag{5.3}$$

The inequality holds because if the term added to the 1 must be small. If it were not small, we would be forced to go beyond linear order in  $Q_f$ , and hence the perturbative approach would no longer be trusted. Therefore it does not change the sign of any of the functions where the perturbative analysis is valid. Hence, the action that we have written had positive definite hamiltonian

$$\int d^5\xi \left[ \mathcal{F}_0(\xi; Q_f) + \mathcal{F}_1(\xi; Q_f) \left( (\partial_t M)^2 + \sum_{i=1}^3 (\partial_{x_i} M)^2 \right) + \mathcal{F}_2(\xi; Q_f) (\partial_{\xi^5} M)^2 + \mathcal{F}_3(\xi; Q_f) M^2 \right] \quad (5.4)$$

simply because it is a sum of squares.

For this reason, what seems to be important is the presence of a cancelation between the DBI and CS action at the order  $|M|$  (we expect such terms from the solutions above).

Of course another interesting question is what happens outside of the regime of validity of the perturbative approach. This, however, is out of the scope of our present investigation, although we hope to address this in some future work.

### 5.1 DBI and CS equations of motion: $x_4 = 0$ ( $\pi R_x$ ) solution

Of course to expand an action, we must expand about some *solution* to the equations of motion. For this reason, we briefly outline (and give more detail in appendix C) why  $x_4 = 0$  ( $x_4 = \pi R_x$ ) is still a solution to the equations of motion. One may address this simply by looking at the gauge invariant information in the equations of motion: namely the cusps. The cusps in  $\hat{\phi}_1, A_1, G_1$  and  $C_1$  cannot be removed with a coordinate transformations, as this would introduce delta functions into  $B_1$ . In appendix C, we show that the cusp in  $B_1$ , which is pure gauge, does not enter. We can read off the behavior around the cusps<sup>11</sup> to be

$$\begin{aligned} \hat{\phi}_1 &= f_{\hat{\phi}_1}(u) + \frac{1}{2} \frac{\sqrt{3}u^{\frac{3}{2}} \left(1 - \frac{U^3}{u^3}\right)^{\frac{1}{2}} |x_4|}{(Q_c g_s)^{\frac{1}{2}}} + \mathcal{O}(x_4^2) \\ F_1 &= f_1(u) - \frac{1}{2} 2 \frac{\sqrt{3}u^{\frac{3}{2}} \left(1 - \frac{U^3}{u^3}\right)^{\frac{1}{2}} |x_4|}{(Q_c g_s)^{\frac{1}{2}}} + \mathcal{O}(x_4^2) \\ F_2 &= f_2(u) + \frac{1}{2} 4 \frac{\sqrt{3}u^{\frac{3}{2}} \left(1 - \frac{U^3}{u^3}\right)^{\frac{1}{2}} |x_4|}{(Q_c g_s)^{\frac{1}{2}}} + \mathcal{O}(x_4^2) \end{aligned} \quad (5.5)$$

simply by comparing the delta function source terms and the coefficient (a function of  $u$ ) of the  $\partial_{x_4}^2$  term. The next order contributions are of order  $x_4^2$  because we require the functions be even about  $x_4 = 0$  to obey the  $\mathbb{Z}_2$  symmetry of the problem.

Now we need to find the equations of motion for the embedding functions  $X^\mu$  resulting from the action

$$-\frac{g_s S_B}{Q_f} = \int d^9\xi e^{-\phi} \sqrt{-g_p} + \int A_9. \quad (5.6)$$

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<sup>11</sup>at  $x_4 = 0$ , similar conditions apply at  $x_4 = \pi R_x$

In the appendix, we show that the only interesting equation of motion is the one for  $X^4$ , and we find that this becomes

$$\begin{aligned}
 & -\sqrt{-g_p} \frac{u^{-\frac{3}{4}}}{g_s R_{D4}^{-\frac{3}{4}}} \partial_{x_4} \left( \frac{u^{\frac{3}{2}} \left(1 - \frac{U_K^3}{u^3}\right)^{\frac{1}{2}} |x_4|}{(R_{D4})^{\frac{3}{2}}} + \mathcal{O}(x_4^2) \right) \\
 & - \pm \partial_{x_4} \left( \sqrt{-g_p} |x_4| \frac{1}{g_s} \frac{u^{\frac{3}{4}} \left(1 - \frac{U_K^3}{u^3}\right)^{\frac{1}{2}}}{R_{D4}^{\frac{3}{4}}} + \mathcal{O}(x_4^2) \right) \\
 & = -\sqrt{-g_p} \frac{u^{\frac{3}{4}} \left(1 - \frac{U_K^3}{u^3}\right)^{\frac{1}{2}}}{R_{D4}^{\frac{3}{4}} g_s} \partial_{x_4} (|x_4| \pm |x_4|)
 \end{aligned} \tag{5.7}$$

where in the third line, we ignore the higher order in  $x_4$  corrections, as we will evaluate the derivative at  $x_4 = 0$ . Above we have also switched back to the more familiar  $g_s Q_c = 3R_{D4}^3$  notation.

The term with the  $\pm$  comes from the CS term, and may vanish for the  $-$  sign choice above. We interpret this as putting a brane next to the backreacted branes. This tells that the equations of motion are satisfied for a brane placed directly on top of the other branes. If instead we had put an anti-brane, we would have found a constant force type potential, and we take that this is a solution too (although we expect an open string tachyon for small enough distances: our actions do not contain terms for strings ending on different branes).

Further, we should note that the above is a gauge independent statement. The cusps in the functions  $A_1, C_1, G_1, A_{(9)}$  are independent of the gauge choice,<sup>12</sup> and these were the only functions that contribute above (see appendix C). Therefore, the leading  $|x_4|$  dependence is unaffected by small coordinate transformations. For this statement, it is important that  $B$  does not appear: its values (cusps and all) are gauge dependent, while the other functions are determined in a gauge covariant way. Recall that while  $\hat{\phi}_1$  is gauge dependent, its cusp behavior is not: one may not remove any part of the cusp without introducing unwanted delta functions into  $B_1$ .

This gives that to lowest order, original embedding solution is still a solution to the equations of motion. To truly consider the stability, however, we would like to know whether this extremum of the action is a maximum or a minimum. For this we will need to investigate the second order action about this point, and this will involve second derivatives in  $x_4$ , rather than just first derivatives. Hence the even functions that we were able to ignore in the above discussion will enter.

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<sup>12</sup>in any sense: small coordinate transformation or shifts of  $A_{(9)}$  by a infinitely differentiable globally exact form

## 5.2 $U_K = 0$ decompactification limit

We begin with the solutions for the decompactified case in the last section

$$\begin{aligned}
 F_i(u, x_4) &= uK_i(q) \quad q = \frac{x_4\sqrt{u}}{(Q_c g_s)^{\frac{1}{2}}} \\
 \hat{\phi}_1(u, x_4) &= uK_3(q),
 \end{aligned}
 \tag{5.8}$$

where  $K_i$  are given in equation (4.5). We will want to construct the second order action in  $x_4$  and so we will need  $K_i$  to second order in  $x_4 \propto q$ . We expand the  $K_i$  to obtain  $F_i$  and  $\hat{\phi}_1$  and find

$$\begin{aligned}
 F_1(u, x_4) &= uK_1(q) = \frac{256}{1001}u - \frac{\sqrt{3}u^{\frac{3}{2}}|x_4|}{(Q_c g_s)^{\frac{1}{2}}} + \frac{768u^2 x_4^2}{143Q_c g_s} \\
 F_2(u, x_4) &= uK_2(q) = -N_2 \frac{\sqrt{3}}{9}u + \frac{2\sqrt{3}u^{\frac{3}{2}}|x_4|}{(Q_c g_s)^{\frac{1}{2}}} + N_2 \frac{2\sqrt{3}u^2 x_4^2}{3Q_c g_s} \\
 \hat{\phi}_1(u, x_4) &= uK_3(q) = N_3 \frac{u}{4} + \frac{1}{2} \frac{\sqrt{3}u^{\frac{3}{2}}|x_4|}{(Q_c g_s)^{\frac{1}{2}}} - N_3 \frac{3u^2 x_4^2}{16Q_c g_s}
 \end{aligned}
 \tag{5.9}$$

where we have dropped order  $\mathcal{O}(x_4^3)$  and higher terms.

We are now able to examine stability of the decompactified limit by examining the second order action. We start by writing the pullback metric as a function of  $X^4(x^\mu, u)$

$$ds_p^2 = e^{2A} (\eta_{\mu\nu} dx^\mu dx^\nu) + e^{2G} du^2 + e^{2C} d\Omega_4^2 + e^{2B} (\partial_\mu X^4 + \partial_u X^4 du)^2.
 \tag{5.10}$$

We need the determinant of this metric to second order in  $X^4$ , however it is easier not to expand the above metric completely, and simply realize that

$$\sqrt{-g_p} = \sqrt{-g_{p0}} \left( 1 + \frac{1}{2} g_{p0}^{ab} h_{ab} \right) + \mathcal{O}((X^4)^4)
 \tag{5.11}$$

where  $g_{p0}$  is constructed by dropping the last term in  $ds_p^2$ , and  $h_{ab}$  is the symmetric tensor defined by the last term in  $ds_p^2$ . Evaluating this, we find

$$\begin{aligned}
 & - \int d^9 \xi e^{-\phi} \sqrt{-g_p} = \\
 & - \int d^9 \xi e^{-\phi + 4A + 4C + G} \left( 1 + e^{2(B-A)} \frac{1}{2} \eta^{\mu\nu} \partial_\mu X^4 \partial_\nu X^4 + e^{2(B-G)} \frac{1}{2} \partial_u X^4 \partial_u X^4 \right)
 \end{aligned}
 \tag{5.12}$$

again, dropping order  $(X^4)^3$  and higher. We now take the expansion of the functions  $A = A_0 + Q_f A_1, \dots$  and evaluate to zeroth and first order in  $Q_f$ , using the above expansions



about  $X^4 = 0$ , and keeping only those terms second order in  $X^4$  or lower.

$$\begin{aligned}
 & - \int d^9 \xi e^{-\phi} \sqrt{-g_p} = \\
 & - \int d^9 \xi \frac{u^4 \left(\frac{Q_c g_s}{3}\right)^{\frac{1}{2}}}{g_s u^{\frac{3}{2}}} \left[ 1 - Q_f \left( \frac{512}{5005} + N_2 \frac{4\sqrt{3}}{45} + N_3 \frac{9}{20} \right) u \right. \\
 & \quad + Q_f \frac{\sqrt{3} u^{\frac{3}{2}} |X^4|}{\sqrt{Q_c g_s}} + Q_f \left( -\frac{1536}{715} + N_2 \frac{8\sqrt{3}}{15} + N_3 \frac{33}{80} \right) \frac{u^2 X_4^2}{Q_c g_s} \quad (5.13) \\
 & \quad + \left( 1 - Q_f \left( \frac{512}{5005} + N_2 \frac{\sqrt{3}}{9} + N_3 \frac{11}{20} \right) u \right) \frac{1}{2} \eta^{\mu\nu} \partial_\mu X^4 \partial_\nu X^4 \\
 & \quad \left. + \frac{3u^3}{Q_c g_s} \left( 1 - Q_f \left( \frac{512}{5005} + N_2 \frac{4\sqrt{3}}{45} + N_3 \frac{9}{20} \right) u \right) \frac{1}{2} \partial_u X^4 \partial_u X^4 \right].
 \end{aligned}$$

To this we must add the term coming from the RR coupling. To do so, we recognize that

$$A_{\mu_i|x_4} = \mp \sqrt{-g_p} \frac{Q_f}{g_s} e^B |x_4| \quad (5.14)$$

satisfies the equations of motion for the nine form potential. We use the notation  $|x_4$  to mean that the index for  $x_4$  has been omitted. The factor of  $-g_p$  in (5.14) is constructed using only the *zereth* order in  $Q_f$  metric pulled back. Further, this  $-g_p$  only has corrections of order  $\mathcal{O}((X^4)^2)$ , and so we may ignore them because of the  $|x_4|$  already multiplying  $\sqrt{-g_p}$ . Hence, to the order that we are working,

$$- \int A_9 = - \int d^9 \xi (\mp) u^4 \frac{Q_f}{g_s} |X^4|. \quad (5.15)$$

Thus, for the correct orientation of the probe brane, the  $|X^4|$  term in the action completely cancels,<sup>13</sup> and the total action becomes

$$\begin{aligned}
 & - \int d^9 \xi e^{-\phi} \sqrt{-g_p} = \\
 & - \int d^9 \xi \frac{u^4 \left(\frac{Q_c g_s}{3}\right)^{\frac{1}{2}}}{g_s u^{\frac{3}{2}}} \left[ 1 - Q_f \left( \frac{512}{5005} + N_2 \frac{4\sqrt{3}}{45} + N_3 \frac{9}{20} \right) u \right. \\
 & \quad + Q_f \left( -\frac{1536}{715} + N_2 \frac{8\sqrt{3}}{15} + N_3 \frac{33}{80} \right) \frac{u^2 X_4^2}{Q_c g_s} \quad (5.16) \\
 & \quad + \left( 1 - Q_f \left( \frac{512}{5005} + N_2 \frac{\sqrt{3}}{9} + N_3 \frac{11}{20} \right) u \right) \frac{1}{2} \eta^{\mu\nu} \partial_\mu X^4 \partial_\nu X^4 \\
 & \quad \left. + \frac{3u^3}{Q_c g_s} \left( 1 - Q_f \left( \frac{512}{5005} + N_2 \frac{4\sqrt{3}}{45} + N_3 \frac{9}{20} \right) u \right) \frac{1}{2} \partial_u X^4 \partial_u X^4 \right].
 \end{aligned}$$

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<sup>13</sup>We interpret this as a brane next to the backreacted brane(s), rather than an anti brane next to the backreacted brane(s).

Although we can at this point find equations of motion for the above action, and proceed with the analysis directly, we find it convenient to manipulate the above equation a bit more. For this, we note that we can redefine the  $u$  coordinate as well as the field  $X^4$ . We find it convenient to do the following transformation

$$\begin{aligned} u &= \hat{u} + Q_f \lambda_1 \hat{u}^2 \\ du &= (1 + 2Q_f \lambda_1 \hat{u}) d\hat{u} \\ X^4(x^\mu, \hat{u}) &= \left(1 + \frac{1}{2} Q_f \lambda_2 \hat{u}\right) \hat{X}^4(x^\mu, \hat{u}) \end{aligned} \tag{5.17}$$

(we do the  $u$  coordinate change first, and then the  $X$  transformation) and then for ease of notation we simply drop the  $\hat{\phantom{x}}$  from the above. Again, we may only keep order  $Q_f$  or lower in the above expansion. After doing this, we will introduce terms of the form  $Q_f \lambda_2 f(u) X^4 \partial_u X^4 = \frac{1}{2} Q_f \lambda_2 f(u) \partial_u ((X^4)^2)$  which we integrate by parts. This affects the coefficient of  $(X^4)^2$ . After doing so, we find that the new action is

$$\begin{aligned} & - \int d^9 \xi e^{-\phi} \sqrt{-g_p} = \\ & - \int d^9 \xi \frac{u^4 \left(\frac{Q_c g_s}{3}\right)^{\frac{1}{2}}}{g_s u^{\frac{3}{2}}} \left[ 1 - Q_f \left( \frac{512}{5005} + N_2 \frac{4\sqrt{3}}{45} + N_3 \frac{9}{20} - \frac{9}{2} \lambda_1 \right) u \right. \\ & \quad + Q_f \left( -\frac{1536}{715} + N_2 \frac{8\sqrt{3}}{15} + N_3 \frac{33}{80} - \frac{11}{8} \lambda_2 \right) \frac{u^2 X_4^2}{Q_c g_s} \\ & \quad + \left( 1 - Q_f \left( \frac{512}{5005} + N_2 \frac{\sqrt{3}}{9} + N_3 \frac{11}{20} - \frac{9}{2} \lambda_1 - \lambda_2 \right) u \right) \frac{1}{2} \eta^{\mu\nu} \partial_\mu X^4 \partial_\nu X^4 \\ & \quad \left. + \frac{3u^3}{Q_c g_s} \left( 1 - Q_f \left( \frac{512}{5005} + N_2 \frac{4\sqrt{3}}{45} + N_3 \frac{9}{20} - \frac{7}{2} \lambda_1 - \lambda_2 \right) u \right) \frac{1}{2} \partial_u X^4 \partial_u X^4 \right]. \end{aligned} \tag{5.18}$$

The interpretation of  $\lambda_1$  and  $\lambda_2$  is that they correspond to coordinate transformations, and so are actually arbitrary and one may choose these. The numbers  $N_1$  and  $N_2$  are numbers that determine part of the profile of the backreaction of the branes, and so may be constrained by some physical boundary conditions. Here, however, we simply note that  $\lambda_2$  may be chosen to eliminate the  $(X^4)^2$  term completely. This then leaves terms of the form  $(1 - Q_f u C_i)$  times terms present when  $Q_f = 0$ . Therefore, we conclude that when the perturbative analysis is valid, all coefficients remain the same as the  $Q_f = 0$  case. Because of this, one may simply argue that the hamiltonian of the above action is positive definite (it is a sum of squares times positive coefficients) for the range of validity of the perturbative analysis. Hence, we conclude that in the perturbative regime, the configuration is stable.

This depended on the leading order cancelation between the DBI and coupling to the RR field. Other than this, the remaining terms were all present in the  $Q_f \rightarrow 0$  limit (up to gauge). In such a case, all corrections that are order  $Q_f$  cannot change the signs of coefficients, and so stability (in the range of validity for the perturbative approach) is preserved. We will see this again in the next section.

As a curious note, with an appropriate choice of  $N_i$  and  $\lambda_i$ , one can completely cancel the leading order in  $Q_f$  contribution to the above action.

### 5.3 Stability of the Sakai Sugimoto model

Above, and in appendix C, we show that  $X_4 = 0, X_4 = \pi R_x$  is still a solution to the equations of motion resulting from the DBI+CS action. To evaluate the second order action for fluctuations, we change to the radial coordinate  $z$  defined by  $u^3 = U_K^3 + U_K z^2$ , and then to the ‘‘Cartesian’’ coordinates

$$y_1 = z \sin\left(\frac{x_4}{R_x}\right) \quad y_2 = z \cos\left(\frac{x_4}{R_x}\right) \quad (5.19)$$

which also allows for comparison with the analysis performed in [10]. Here the important point is that we chose the gauge  $B_1 = G_1$ , and so the only change in the  $z, x_4$  plane is by a conformal factor. Hence, much of the analysis of [10] follows through. We find that the metric in these coordinates is written

$$ds^2 = e^{2A_1} \left(\frac{u}{R_{D4}}\right)^{\frac{3}{2}} (\eta_{\mu\nu} dx^\mu dx^\nu) + e^{2C_1} R_{D4}^{\frac{3}{2}} u^{\frac{1}{2}} d\Omega_4^2 \quad (5.20)$$

$$+ e^{2G_1} \frac{4}{9} \left(\frac{R_{D4}}{u}\right)^{\frac{3}{2}} ([1 - h(z)y_1^2] dy_1^2 + [1 - h(z)y_2^2] dy_2^2 - 2h(z)y_1 y_2 dy_1 dy_2).$$

where now all metric functions are written as functions of  $y_1$  and  $y_2$ , and we have defined the following functions

$$h(z) = \frac{1}{z^2} \left(1 - \frac{U_K}{u}\right)$$

$$u = u(z) = (U_K^3 + U_K z^2)^{\frac{1}{3}} \quad (5.21)$$

$$z = z(y_1, y_2) = \sqrt{y_1^2 + y_2^2}.$$

Here we have suppressed the factor of  $Q_f$  for ease of notation, and will only reintroduce it at the end. Taking the embedding  $y_1(x^\mu, u)$  one may compute the second order action the same way as the  $U_K = 0$  decompactification piece. One writes the line element as  $ds^2 = ds_1^2 + ds_2^2$  where  $ds_1^2$  is diagonal and  $ds_2^2$  is already order  $y_1^2$ , and so again one finds that

$$\sqrt{-g_p} = \sqrt{-g_{p0}} \left(1 + \frac{1}{2} g_{p0}^{ab} h_{ab}\right) + \mathcal{O}((y_1)^4). \quad (5.22)$$

One may compute the DBI action easily now,

$$-K_8 \int d^9 \xi e^{-\phi} \sqrt{-g_p} = \quad (5.23)$$

$$-\frac{2}{3} \frac{K_8 R_{D4}^{\frac{3}{2}} U_K^{\frac{1}{2}} V_4}{g_s} \int d^5 \xi \left( u^2 e^{4A_1 + 4C_1 + G_1 - \phi_1} + \frac{2}{9} \frac{R_{D4}^3}{u} e^{2A_1 + 4C_1 + 3G_1 - \phi_1} \eta^{\mu\nu} \partial_\mu y_1 \partial_\nu y_1 \right.$$

$$\left. + \frac{1}{2} \frac{u^3}{U_K} e^{4A_1 + 4C_1 + G_1 - \phi_1} (h(z)(y_1^2 - 2y_1 y_2 y_1) + y_1^2) \right)$$

where we have defined  $\partial_{y_2} y_1 = y_1$ . Further, the above function  $u(y_1, y_2)$  still must be expanded in  $y_1$ . The term  $V_4$  is the volume of the unit four sphere, and  $d^5\xi = dt dx_1 dx_2 dx_3 dy_2$ .

We will integrate the term linear in  $y_1$  by parts, but first we find it convenient to introduce the following notation

$$\mathcal{A} \equiv 4A_1 + 4C_1 + G_1 - \phi_1 = 2A_1 + 2C_1 + G_1 - \frac{1}{2}\hat{\phi}_1 \tag{5.24}$$

$$\mathcal{B} \equiv 2A_1 + 4C_1 + 3G_1 - \phi_1 = 2C_1 + 3G_1 - \frac{1}{2}\hat{\phi}_1. \tag{5.25}$$

From the arguments in the last subsection, we expect the above combinations of fields to have the following behavior about  $y_1 = 0$

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_0(y_2) + \frac{2}{3}|y_1| + \mathcal{A}_2(y_2)y_1^2 + \dots \\ \mathcal{B} &= \mathcal{B}_0(y_2) + \dots \end{aligned} \tag{5.26}$$

where in  $\mathcal{B}$  we ignore higher corrections in  $y_1$  because its coefficient is already  $\mathcal{O}(y_1^2)$ . In the above, we have determined the expansion in  $\mathcal{A}$  of order  $|y_1|$  by considering the argument in the  $u, x_4$  coordinates used to give equations (5.5), and then changing coordinates to the  $y_1, y_2$  variables.

In the following, we will have to evaluate  $u(y_1, y_2)$  at  $y_1 = 0$ , and henceforth, we will call this function  $u_y$ . Similarly we define  $h_y = h(z)|_{y_1=0}$ . Plugging in the above to the second order action, and reintroducing  $Q_f$  we find

$$\begin{aligned} -K_8 \int d^9\xi e^{-\phi} \sqrt{-g_p} = & \tag{5.27} \\ -\tilde{T} \int d^5\xi \left[ (u_y^2 + y_1^2)(1 + Q_f \mathcal{A}_0) + \frac{2 R_{D4}^3}{9 u_y} (1 + Q_f \mathcal{B}_0) \eta^{\mu\nu} \partial_\mu y_1 \partial_\nu y_1 \right. \\ & \left. + \frac{1}{2} \frac{u_y^3}{U_K} (1 + Q_f \mathcal{A}_0) \dot{y}_1^2 \right. \\ & \left. + Q_f \left( u_y^2 \mathcal{A}_2 + \frac{1}{2} \frac{u_y^3}{U_K} h_y y_2 \partial_{y_2} \mathcal{A}_0 \right) y_1^2 + Q_f u_y^2 \frac{2}{3} |y_1| \right] \end{aligned}$$

where we define

$$\frac{2}{3} \frac{K_8 R_{D4}^{\frac{3}{2}} U_K^{\frac{1}{2}} V_4}{g_s} \equiv \tilde{T} \tag{5.28}$$

as in the work of [10]. It is easy to read off the result in [10] in the  $Q_f = 0$  limit; it is the top two lines of the right hand side. As in the last sections, we now add to this the contribution from  $\int A_0$ . This is relatively easy to do, as we find

$$\sqrt{-g} = \frac{4}{9} R_{D4}^{\frac{3}{2}} U_K^{\frac{1}{2}} u_y^2 \left( 1 + \frac{2}{3} \frac{y_1^2}{u_y^3} + \mathcal{O}(y_1^4) \right) \sqrt{g_{S^4}} \tag{5.29}$$

and so to second order in  $y_1$  we can simply take

$$A_{\mu_i|y_1} = \mp \frac{Q_f}{g_s} \frac{4}{9} R_{D4}^{\frac{3}{2}} U_K^{\frac{1}{2}} u_y^2 |y_1| (1 + \mathcal{O}(y_1^2)) \sqrt{g_{S^4}}. \tag{5.30}$$

This exactly cancels the  $|y_1|$  term (for the  $-$  choice), as we have seen several times now (see appendix C for this occurring at the level of the equations of motion). Therefore, the full action reads

$$\begin{aligned}
 -K_8 \int d^9 \xi e^{-\phi} \sqrt{-g_p} - K_8 \int A_9 = \\
 -\tilde{T} \int d^5 \xi \left[ (u_y^2 + y_1^2)(1 + Q_f \mathcal{A}_0) + \frac{2 R_{D4}^3}{9 u_y} (1 + Q_f \mathcal{B}_0) \eta^{\mu\nu} \partial_\mu y_1 \partial_\nu y_1 \right. \\
 \left. + \frac{1}{2} \frac{u_y^3}{U_K} (1 + Q_f \mathcal{A}_0) \dot{y}_1^2 + Q_f \left( u_y^2 \mathcal{A}_2 + \frac{1}{2} \frac{u_y^3}{U_K} h_y y_2 \partial_{y_2} \mathcal{A}_0 \right) y_1^2 \right].
 \end{aligned} \tag{5.31}$$

At this point it is sufficient to note that because all terms in the action were present before the perturbation, we expect that wherever the perturbative analysis is valid, the stability of the Sakai Sugimoto model is maintained. This is because whenever the perturbation is small, the above action yields a positive definite hamiltonian, and so all fluctuations will have positive energy. Further, as we have seen in the previous section, the equations admit a perturbative solution about  $u = U_K$  and so the perturbative analysis is valid from  $u = U_K$  up to  $u \ll \frac{1}{Q_f}$ , where for the previous sections analysis gives a good approximation to the solutions.

## 6. Discussion and outlook

Here we will summarize our results. From the above calculations, we can see that when  $u$  is large enough, the solutions tend to that of the decompactified case. The height of these functions all grow as  $u$  and so to stay in the perturbative regime, we require that

$$u Q_f \ll 1 \rightarrow u \ll \frac{1}{Q_f} = \frac{4\pi \ell_s}{g_s N_f} \rightarrow u^3 \ll \frac{64\pi^3 \ell_s^3}{g_s^3 N_f^3} \tag{6.1}$$

There is a further requirement, that the supergravity approximation is valid. This gives a restriction

$$g_s \left( \frac{u^3}{R_{D4}^3} \right)^{\frac{1}{4}} \ll 1 \rightarrow u^3 \ll \frac{\pi N_c \ell_s^3}{g_s^3}. \tag{6.2}$$

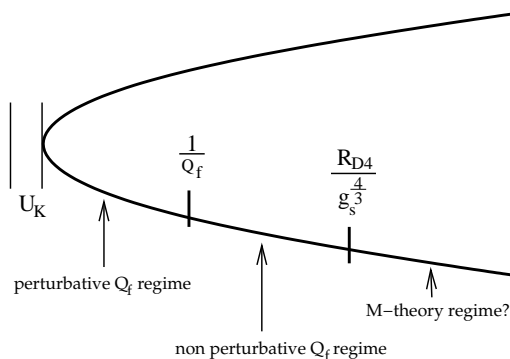
One may easily compare now and see which condition is more stringent, as all coefficients of  $g_s$  and  $\ell_s$  are the same. We find that generically

$$\frac{1}{Q_f^3} < \frac{R_{D4}^3}{g_s^4} \rightarrow 1 < \frac{N_c N_f^3}{64\pi^2} \tag{6.3}$$

as we assume that  $N_c$  is large and  $N_f \neq 0$  (see figure 7). However, we note that for small  $g_s$  the regime of validity of the perturbative backreaction (and so the validity of the probe approximation) becomes arbitrarily large.

In figure 7, we have indicated a possible M-theory (11D SUGRA) lift, although some caution is necessary, as no know lift D8 branes is understood in the context of 11D SUGRA, at least for those described by the Romans type IIA.

We are also left with some obvious open questions:



**Figure 7:** Range of validity for calculations

1. The topic of this paper has been the low temperature limit of the Sakai Sugimoto model, and one may wish to know the qualitative differences between the low and high temperature limits. Further, one may hope that the analysis of the high temperature limit may be easier, as the D8 branes are transverse to a cylinder, rather than a cigar.
  
2. It would be interesting to address the backreaction of flavor branes in other brane systems using the above techniques. While one may worry about the perturbation breaking down near the brane for codimension other than 1, one may trust the cancellation between the DBI and CS terms in the quadratic action for fluctuations. We believe this to be true for the following reason: for a section of brane near a smooth point in a manifold, its backreaction (non perturbative contributions included) should behave just as the flat space case. In such a situation, a section of parallel probe brane near by feels no force on it because the charge and mass (per unit p volume) are the same. We may expect this to always be true. Further, in supersymmetric situations, the supersymmetry of the backreaction may be of some assistance in fixing all coefficients. We look forward to addressing these issues in some future work.

### Acknowledgments

We are grateful for discussions with Oren Bergman, who was involved in the initial stages of this project. We would like to thank Ofer Aharony for discussions and comments on the draft of this paper. We also wish to thank Martha Merzig for help editing graphs, and Josh Davis for directing us to a useful reference. The work of B.B and J. S has been supported in part by the Israel Science Foundation, by a grant ( DIP H.52) of German Israel Project Cooperation D.I.P and by the European Network MRTN-CT-2004-512194.

This material is based upon work supported by the National Science Foundation under Grant No. PHY-0455649.

## A. Massive type IIA equations of motion

We recall the following definitions

$$\begin{aligned}
 \tilde{F}_2 &= F_2 + MB_2 \\
 \hat{F}_4 &= F_4 + \frac{1}{2}MB_2 \wedge B_2 \\
 \tilde{F}_4 &= F_4 - A_1 \wedge H_3 + \frac{1}{2}MB_2 \wedge B_2.
 \end{aligned} \tag{A.1}$$

The equations of motion for the action  $S_{IIAM}$  (2.9) are

$$R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi - \frac{1}{2!}H_\mu{}^{\rho\sigma}H_{\nu\rho\sigma} - e^{2\phi} \frac{1}{2 \cdot 2!} \left( 2\tilde{F}_\mu{}^\rho \tilde{F}_{\nu\rho} - \frac{1}{2}g_{\mu\nu} \tilde{F}_2 \cdot \tilde{F}_2 \right) \tag{A.2}$$

$$\begin{aligned}
 &+ e^{2\phi} \frac{1}{2 \cdot 4!} \left( 4\tilde{F}_\mu{}^{\rho_1\rho_2\rho_3} \tilde{F}_{\nu\rho_1\rho_2\rho_3} - \frac{1}{2}g_{\mu\nu} \tilde{F}_4 \cdot \tilde{F}_4 \right) + \frac{e^{2\phi}}{2}g_{\mu\nu}M^2=0 \\
 &R - 4\partial\phi \cdot \partial\phi - \frac{1}{2 \cdot 3!}H_3 \cdot H_3 + 4g^{\mu\nu} \nabla_\mu \nabla_\nu \phi=0 \tag{A.3}
 \end{aligned}$$

$$\begin{aligned}
 \nabla_{\alpha_1} \left( e^{-2\phi} H^{\alpha_1\beta_1\beta_2} + A_{\alpha_2} \tilde{F}^{\alpha_1\alpha_2\beta_1\beta_2} \right) - \frac{1}{2! \cdot 4! \cdot 4! \sqrt{-g}} \epsilon^{\beta_1\beta_2\cdots} F_{(4)\dots} F_{(4)\dots} \\
 - M \tilde{F}^{\beta_1\beta_2} - \frac{1}{2} M B_{\alpha_1\alpha_2} \tilde{F}^{\alpha_1\alpha_2\beta_1\beta_2} - \frac{1}{2 \cdot 2! \cdot 4! \sqrt{-g}} \epsilon^{\beta_1\beta_2\cdots} B \dots B \dots \hat{F}_{(4)\dots} = 0 \tag{A.4}
 \end{aligned}$$

$$\nabla_{\alpha_1} \tilde{F}^{\alpha_1\beta_1} - \frac{1}{3!} \tilde{F}^{\beta_1\alpha_2\alpha_3\alpha_4} H_{\alpha_2\alpha_3\alpha_4} = 0 \tag{A.5}$$

$$\begin{aligned}
 \nabla_{\alpha_1} \tilde{F}^{\alpha_1\beta_1\beta_2\beta_3} - \frac{1}{3! \cdot 4! \sqrt{-g}} \epsilon^{\beta_1\beta_2\beta_3\cdots} H \dots \hat{F}_{(4)\dots} \\
 - \frac{M}{4! \sqrt{-g}} \epsilon^{\beta_1\beta_2\beta_3\cdots} B \dots B \dots H \dots = 0 \tag{A.6}
 \end{aligned}$$

$$\begin{aligned}
 &-\frac{1}{2} B \cdot \tilde{F}_2 - \frac{3}{4!} B \dots B \dots \tilde{F}_4 \\
 &-\frac{3}{2! \cdot 4! \cdot 4! \sqrt{-g}} \epsilon^{\cdots} B \dots B \dots B \dots \tilde{F}_{(4)\dots} - M + *F_{10}=0 \\
 &dM=0 \tag{A.7}
 \end{aligned}$$

where again  $\epsilon$  takes values  $\pm 1$ . In the above, where we have written  $\cdots$  there are indices contracted. To reintroduce the (sub)superscripts, one puts in a set of indices in the superscripts, and then puts the same indices in the same order in the subscripts. In the above, we note that  $M$  is constant, and may be considered piecewise constant in the presence of sources.

## B. Separating the equations

Here we deal with the Einstein equations and the dilaton equation, and explain how to

separate them. There are 5 Einstein equations, and one for the dilaton, and we name them

$$\begin{aligned}
EOM_{\hat{\phi}} = & 2\partial_u^2 \hat{\phi} - \left(\partial_u \hat{\phi}\right)^2 - 2\partial_u^2 B - 4(\partial_u A)^2 - 2(\partial_u B)^2 - 4(\partial_u C)^2 \\
& + 2\partial_u \hat{\phi} \partial_u B - 2\partial_u \hat{\phi} \partial_u G + 2\partial_u B \partial_u G \\
& + e^{(2G-2B)} \left( 2\partial_{x_4}^2 \hat{\phi}(u, x_4) - \left(\partial_{x_4} \hat{\phi}\right)^2 - 4(\partial_{x_4} A)^2 - 2(\partial_{x_4} G)^2 - 4(\partial_{x_4} C)^2 \right. \\
& \left. + 2\partial_{x_4} \hat{\phi} \partial_{x_4} G - 2\partial_{x_4} \hat{\phi} \partial_{x_4} G + 2\partial_{x_4} B \partial_{x_4} G - 2\partial_{x_4}^2 G \right) \tag{B.1}
\end{aligned}$$

$$\begin{aligned}
& + 12e^{2G-2C} - \frac{e^{2A-B+2G+2C+\frac{1}{2}\hat{\phi}} Q_f \Delta}{g_s} \\
EOM_{tt} = & 4e^{(G-B+\hat{\phi})} \partial_u \left( e^{(-G+B-\hat{\phi})} \partial_u A \right) \\
& + 4e^{(2G-2B)} e^{(B-G+\hat{\phi})} \partial_{x_4} \left( e^{(-B+G-\hat{\phi})} \partial_{x_4} A \right) \tag{B.2} \\
& - Q_c^2 e^{4A+2G-4C+\hat{\phi}} - 2 \frac{e^{2A-B+2G+2C+\frac{1}{2}\hat{\phi}} Q_f \Delta}{g_s}
\end{aligned}$$

$$\begin{aligned}
EOM_{xx} = & 4e^{(-B+G+\hat{\phi})} \partial_u \left( e^{(B-G-\hat{\phi})} \partial_u B \right) \\
& + 4e^{(2G-2B)} \left( 4(\partial_{x_4} A)^2 + 4(\partial_{x_4} C)^2 + (\partial_{x_4} G)^2 + e^{(B)} \partial_{x_4} \left( e^{(-B)} \partial_{x_4} (G - \hat{\phi}) \right) \right) \\
& - Q_c^2 e^{4A+2G-4C+\hat{\phi}} + 2 \frac{e^{2A-B+2G+2C+\frac{1}{2}\hat{\phi}} Q_f \Delta}{g_s} \tag{B.3}
\end{aligned}$$

$$\begin{aligned}
EOM_{uu} = & 4 \left( 4(\partial_u A)^2 + 4(\partial_u C)^2 + (\partial_u B)^2 + e^{(G)} \partial_u \left( e^{(-G)} \partial_u (B - \hat{\phi}) \right) \right) \\
& + 4e^{(2G-2B)} e^{(-G+B+\hat{\phi})} \partial_{x_4} \left( e^{(G-B-\hat{\phi})} \partial_{x_4} G \right) \\
& - Q_c^2 e^{4A+2G-4C+\hat{\phi}} - 2 \frac{e^{2A-B+2G+2C+\frac{1}{2}\hat{\phi}} Q_f \Delta}{g_s} \tag{B.4}
\end{aligned}$$

$$\begin{aligned}
EOM_{ss} = & -4e^{(G-B+\hat{\phi})} \partial_u \left( e^{(-G+B-\hat{\phi})} \partial_u C \right) - 4e^{(2G-2B)} e^{(B-G+\hat{\phi})} \partial_{x_4} \left( e^{(-B+G-\hat{\phi})} \partial_{x_4} C \right) \\
& + 12e^{(2G-2C)} - Q_c^2 e^{4A+2G-4C+\hat{\phi}} + 2 \frac{e^{2A-B+2G+2C+\frac{1}{2}\hat{\phi}} Q_f \Delta}{g_s} \tag{B.5}
\end{aligned}$$

$$EOM_m = -4\partial_{x_4} A \partial_u A - 4\partial_{x_4} C \partial_u C + \partial_{x_4} \partial_u \hat{\phi} - \partial_u B \partial_{x_4} \hat{\phi} - \partial_{x_4} G \partial_u \hat{\phi} \tag{B.6}$$

Each of the above equations is to be set to zero. The labeling we have used is that the subscript  $\hat{\phi}$  denotes to the  $\phi$  equation of motion,  $tt$  denotes the time-time and  $x_i x_i$  Einstein equations (these are just one equation),  $xx$  denotes the  $x_4 x_4$  equation of motion,  $uu$  the  $uu$ ,  $ss$  the directions along the sphere, and  $m$  the mixed  $ux_4$  equation. Further, we have taken  $\Delta = \Delta(x_4)$  to be a function only of  $x_4$ .

We now wish to perturb the following equations about the background solution. For



this purpose, we take the following expansion

$$\begin{aligned}
A(u, x_4) &= \frac{3}{4} \ln \left( \frac{3^{(\frac{1}{3})} u}{Q_c^{(\frac{1}{3})} g_s^{(\frac{1}{3})}} \right) + Q_f A_1(u, x_4) \\
B(u, x_4) &= \frac{3}{4} \ln \left( \frac{3^{(\frac{1}{3})} u}{Q_c^{(\frac{1}{3})} g_s^{(\frac{1}{3})}} \right) + \frac{1}{2} \ln \left( 1 - \frac{U_K^3}{u^3} \right) + Q_f B_1(u, x_4) \\
G(u, x_4) &= -\frac{3}{4} \ln \left( \frac{3^{(\frac{1}{3})} u}{Q_c^{(\frac{1}{3})} g_s^{(\frac{1}{3})}} \right) - \frac{1}{2} \ln \left( 1 - \frac{U_K^3}{u^3} \right) + Q_f G_1(u, x_4) \\
C(u, x_4) &= -\frac{3}{4} \ln \left( \frac{3^{(\frac{1}{3})} u}{Q_c^{(\frac{1}{3})} g_s^{(\frac{1}{3})}} \right) + \ln(u) + Q_f C_1(u, x_4) \\
\hat{\phi}(u, x_4) &= \frac{3}{2} \ln \left( \frac{3^{(\frac{1}{3})} u}{Q_c^{(\frac{1}{3})} g_s^{(\frac{1}{3})}} \right) - 4 \ln(u) + 2 \ln(g_s) + Q_f \hat{\phi}_1(u, x_4)
\end{aligned} \tag{B.7}$$

It is now straightforward (and rather unilluminating) to expand the equations of motion and keep only the linear term in  $Q_f$ . The only key point is that the source is already linear in  $Q_f$  and so one plugs in the background fields only to the exponential  $e^{2A-B+2G+2C+\frac{1}{2}\hat{\phi}}$  appearing with  $\Delta$ . Rather than writing this out explicitly, we will simply explain the steps involved needed to separate the equations. Henceforth when we write  $EOM_i$  we mean the above equation of motion expanded to linear order in  $Q_f$ . First, the most useful equation when expanded is equation  $EOM_m$  (B.6), and this becomes

$$-\frac{1}{u} \partial_{x_4} C_1 + \frac{5}{2} \frac{1}{u} \partial_{x_4} G_1 - \frac{3}{u} \partial_{x_4} A_1 + \partial_{x_4} \partial_u \hat{\phi}_1 - \frac{3}{4} \partial_{x_4} \hat{\phi}_1 \frac{(u^3 + U_K^3)}{u(u^3 - U_K^3)} = 0. \tag{B.8}$$

This may be integrated to give

$$-\frac{1}{u} C_1 + \frac{5}{2} \frac{1}{u} G_1 - \frac{3}{u} A_1 + \partial_u \hat{\phi}_1 - \frac{3}{4} \hat{\phi}_1 \frac{(u^3 + U_K^3)}{u(u^3 - U_K^3)} + F(u) = 0. \tag{B.9}$$

One may solve this for  $G_1$  and plug into the other equations. We will denote doing so as  $EOM_i|_{G_1}$ . One may easily solve for  $F(u)$  now,

$$2EOM_{\hat{\phi}}|_{G_1} + EOM_{xx}|_{G_1} = -4 \frac{(\partial_u F(u) u^4 + 4F(u) u^3 - U_K^3 \partial_u F(u) u - F(u) U_K^3)}{u(u^3 - U_K^3)} \tag{B.10}$$

and so

$$F(u) = \frac{C_F}{u(u^3 - U_K^3)}. \tag{B.11}$$

However one can easily see that this perturbation is simply taking  $U_K \rightarrow U_K + \delta U_K$  and linearizing on  $\delta U_K$ . This is because under this shift, neither  $\hat{\phi}_1$  nor  $A_1$  nor  $C_1$  is changed, and so only  $G_1$  shifts in  $EOM_m$ . The linear shift of  $G$  is given by  $\frac{3U_K^2 \delta U_K}{u^3 - U_K^3}$ . Thus, we may safely absorb  $F(u)$  into a shift into the definition of  $U_K$ . If need be, we may always reintroduce it by shifting equations that depend on  $B_1$  or  $G_1$  appropriately. Further, this is only a zero mode contribution (in  $x_4$ ) and so will leave unaffected much of our discussion.

For these reasons, we take  $F(u) = 0$  for the time being, knowing how to reintroduce it later if need be.

At this point we have eliminated 2 equations of motion at the cost of 1 function, which puts us on course to decouple the equations.

Next, we make the simple observation that in all equations of motion  $EOM_i|_{G_1}$  only  $\partial_u B_1$  and  $\partial_u^2 B_1$  appear. Thus, if we can solve for  $\partial_u B_1$ , we may eliminate  $B_1$  completely. We do so by taking  $EOM_{xx}|_{G_1} - EOM_{uu}|_{G_1}$  and solving this for  $\partial_u B_1$ . This combination still has a delta function, and so it is important at this step that  $\Delta$  is a function only of  $x_4$  so that when the expression for  $\partial_u B_1$  is substituted into  $\partial_u \partial_u B_1$  no derivatives of delta functions appear.

We have now eliminated 3 of the 6 total equations, with the remaining combinations being  $EOM_{tt}|_{G_1, \partial_u B_1}$ ,  $EOM_{xx}|_{G_1, \partial_u B_1} = \frac{1}{2}EOM_{\hat{\phi}}|_{G_1, \partial_u B_1} = EOM_{uu}|_{G_1, \partial_u B_1}$ ,  $EOM_{ss}|_{G_1, \partial_u B_1}$ . However, here we find that

$$\frac{2}{5}EOM_{tt}|_{G_1, \partial_u B_1} - \frac{1}{3}EOM_{xx}|_{G_1, \partial_u B_1} - \frac{2}{15}EOM_{ss}|_{G_1, \partial_u B_1} = 0 \quad (\text{B.12})$$

Hence, we are left with only 2 independent equations for 3 unknown functions. This appears to be under constrained, however, these equations are actually equations only of 2 linear combinations of the 3 functions. The decoupled combinations may be written

$$-3EOM_{tt}|_{G_1, \partial_u B_1} - \frac{3}{2}EOM_{ss}|_{G_1, \partial_u B_1} = 3\partial_u^2 F_1 + \frac{3(4u^3 - U_K^3)\partial_u F_1}{u(u^3 - U_K^3)} + \frac{g_s Q_c u^3 \partial_{x_4}^2 F_1}{(u^3 - U_K^3)^2} \quad (\text{B.13})$$

$$- \frac{54uF_1}{(u^3 - U_K^3)} + \frac{2u(Q_c g_s)^{\frac{2}{3}} \sqrt{3}}{(u^3 - U_K^3)} \sqrt{\frac{u}{(Q_c g_s)^{\frac{1}{3}} \left(1 - \frac{U_K^3}{u^3}\right)}} \Delta$$

$$\frac{3}{2}EOM_{tt}|_{G_1, \partial_u B_1} - 3EOM_{ss}|_{G_1, \partial_u B_1} = 3\partial_u^2 F_2 + \frac{3(4u^3 - U_K^3)\partial_u F_2}{u(u^3 - U_K^3)} + \frac{g_s Q_c u^3 \partial_{x_4}^2 F_2}{(u^3 - U_K^3)^2}$$

$$- \frac{4u(Q_c g_s)^{\frac{2}{3}} \sqrt{3}}{(u^3 - U_K^3)} \sqrt{\frac{u}{(Q_c g_s)^{\frac{1}{3}} \left(1 - \frac{U_K^3}{u^3}\right)}} \Delta \quad (\text{B.14})$$

where

$$A_1 = -\frac{1}{5}F_1 + \frac{1}{10}F_2 - \frac{3}{10}\hat{\phi}_1 \quad (\text{B.15})$$

$$C_1 = \frac{1}{10}F_1 + \frac{1}{5}F_2 - \frac{1}{10}\hat{\phi}_1. \quad (\text{B.16})$$

We now turn to the question of fixing  $\hat{\phi}$ . For this purpose, we remember that we used the combination

$$\begin{aligned} EOM_{xx} - EOM_{uu} = & \\ & 4\partial_u^2 \hat{\phi}_1 - \frac{4}{3} \frac{u^3 Q_c g_s \partial_{x_4}^2 \hat{\phi}_1}{(u^3 - U_K^3)^2} - \frac{2(4\partial_u C_1 - 5\partial_u G_1 - 5\partial_u B_1 + 12\partial_u A_1)}{u} \\ & + \frac{\frac{4}{3}u(Q_c g_s)^{\frac{2}{3}} \sqrt{3}}{(u^3 - U_K^3)} \sqrt{\frac{u}{(Q_c g_s)^{\frac{1}{3}} \left(1 - \frac{U_K^3}{u^3}\right)}} \Delta \end{aligned} \quad (\text{B.17})$$

to solve for  $\partial_u B_1$ . It is clear for our setup that  $\hat{\phi}$  must have some “kink” part in its solution to account for the delta function, as the only  $x_4$  derivatives that appear act on  $\hat{\phi}$ . However, by adding a zero  $-8/u\partial_u(u \int EOM_m dx_4)$  to the above expression, we find

$$\begin{aligned}
 EOM_{xx} - EOM_{uu} - \frac{8}{u}\partial_u(u \int EOM_m, dx_4) = \\
 -4\partial_u^2 \hat{\phi}_1 - \frac{4}{3} \frac{u^3 Q_c g_s \partial_{x_4}^2 \hat{\phi}_1}{(u^3 - U_K^3)^2} - \frac{2(u^3 - 7U_K^3) \partial_u \hat{\phi}_1}{u(u^3 - U_K^3)} - \frac{36uU_K^3 \hat{\phi}_1}{(u^3 - U_K^3)^2} \\
 + \frac{4}{3} \frac{u(Q_c g_s)^{\frac{2}{3}} \sqrt{3}}{(u^3 - U_K^3)} \sqrt{\frac{u}{(Q_c g_s)^{\frac{1}{3}} \left(1 - \frac{U_K^3}{u^3}\right)}} \Delta + \frac{10}{u} (\partial_u B_1 - \partial_u G_1) \quad (B.18)
 \end{aligned}$$

Now it becomes clear how one may maintain continuity of the functions and at the same time separate the equations. We take  $B_1 = G_1$  and then solve the remaining equation above. One may have guessed this gauge, as one can bring any two dimensional metric to a conformally flat one. We do not impose this on the full metric, however, as the polynomials in  $u$  are easier to work with.

One more comment is in order. If one wishes, one may linearize on a small change  $U_K \rightarrow U_K + \delta U_K$ . Under this,  $B_1$  and  $G_1$  transform differently. This can give a new source term to the equation for  $\hat{\phi}_1$ . However, this change only affects the zero mode (in  $x_4$ ) of  $\hat{\phi}_1$ , and hence will not affect the shape of  $\hat{\phi}_1$  in the  $x_4$  direction.

### C. EOM for DBI+CS: details of $x^4 = 0$ solution

Here we find the equations of motion for the embedding functions  $X^\mu$  resulting from the action

$$-\frac{g_s S_B}{Q_f} = \int d^9 \xi e^{-\phi} \sqrt{-g_p} + \int A_9, \quad (C.1)$$

and explicitly show that  $x_4 = 0$  ( $\pi R_x$ ) is still a solution. We will find the equations of motion for the first part, and then turn our attention to the second part of the above action. First, we change frame by scaling the metric  $g_{\text{string}} = \exp(\frac{2}{9}\phi)G$  to write the first part of the action

$$S_D \equiv \int d^9 \xi \sqrt{-G_p} \quad (C.2)$$

The equations of motion for the fields  $X^\mu(\xi)$  in the above action are

$$\frac{\delta S_D}{\delta X^\mu} = -\sqrt{-G_p} G_{\mu\nu} \left( \nabla_p^2 X^\nu + G_p^{ab} \frac{\partial X^\alpha}{\partial \xi^a} \frac{\partial X^\beta}{\partial \xi^b} \Gamma_{\alpha\beta}^\nu \right) \quad (C.3)$$

where objects with a  $p$  subscript are constructed using the pullback metric, and  $\Gamma$  is the full spacetime Christoffel connection. We wish to ask whether  $X^i = \xi^i$  for  $i \neq x_4$  and  $X^4 = x_4 = \text{constant}$  is a solution to the equations of motion. Consider first the  $X^k$ :

$$\begin{aligned}
 \frac{\delta S_D}{\delta X^k} = -\sqrt{-G_p} G_{ki} \left( \frac{1}{\sqrt{-G_p}} \partial_a G_p^{ab} \sqrt{-G_p} \partial_b X^i \right. \\
 \left. + G_p^{ab} \partial_a X^\alpha \partial_b X^\beta \frac{1}{2} G^{i\rho} (G_{\rho\alpha,\beta} + G_{\rho\beta,\alpha} - G_{\alpha\beta,\rho}) \right) \quad (C.4)
 \end{aligned}$$

where we use  $\partial_a$  as shorthand for a partial derivative in  $\xi^a$ , and we have used the fact that our metric is diagonal. Choosing the  $X^i = \xi^i$  removes the derivative from the first part of the equation. Also, the fact that  $G$  is diagonal, and identical to the pullback metric for indices  $i, j$ , allows us to simplify the above further

$$= -\sqrt{-G_p} G_{(p)ki} \left( \frac{1}{\sqrt{-G_p}} \partial_i G_p^{ii} \sqrt{-G_p} + G_p^{ab} \partial_a X^\alpha \partial_b X^\beta \frac{1}{2} G_p^{ii} (\delta_{\alpha i} G_{ii,N} + \delta_{\beta i} G_{ii,M} - G_{\alpha\beta,i}) \right) \quad (C.5)$$

where  $i$  inside the parentheses are not summed. Note that the  $\delta_{N_i}$  projects the last remaining metric down to the pullback metric, and we are left with

$$= -\sqrt{-G_p} G_{(p)ki} \left( \frac{1}{\sqrt{-G_p}} \partial_i G_p^{ii} \sqrt{G_p} + G_p^{ii} G_p^{ii} G_{ii,i} - \frac{1}{2} G_p^{ii} G_p^{ab} G_{ab,i} \right) \quad (C.6)$$

which simplifies further to

$$= -\sqrt{-G_p} G_{(p)ki} \left( G_{(p),i}^{ii} + \frac{1}{2} G_p^{ii} G_p^{ab} G_{(p)ab,i} + G_p^{ii} G_p^{ii} G_{(p)ii,i} - \frac{1}{2} G_p^{ii} G_p^{ab} G_{ab,i} \right). \quad (C.7)$$

This is obviously zero: the metric is diagonal, so that

$$G_p^{ii} G_{(p)ii} = 1 \rightarrow G_{(p),i}^{ii} G_{(p)ii} + G_p^{ii} G_{(p)ii,i} = 0 \quad (C.8)$$

which then causes the first and third terms to cancel.

Hence, we are left with evaluating the  $X^4$  equation of motion:

$$-\sqrt{-G_p} G_{44} \left( \nabla_p^2 X^4 + G_p^{ab} \partial_a X^\mu \partial_b X^\nu \Gamma_{\mu\nu}^4 \right) \quad (C.9)$$

where we use the shorthand 4 to mean the  $x_4$  components.

The first term vanishes as  $X_4 = \text{constant}$ . The second term we evaluate similarly to the last discussion

$$\begin{aligned} & -\sqrt{-G_p} G_{44} \left( \nabla_p^2 X^4 + G_p^{ab} \partial_a X^L \partial_b X^M \Gamma_{LM}^4 \right) \\ &= -\sqrt{-G_p} G_{44} \left( G_p^{ab} \partial_a X^\mu \partial_b X^\nu \frac{1}{2} G^{44} (\delta_{\mu 4} G_{44,\nu} + \delta_{\nu 4} G_{44,\mu} - G_{\mu\nu,4}) \right). \end{aligned} \quad (C.10)$$

This time, however, the  $\delta_{M4}$  and  $\delta_{N4}$  give zero (as the  $X^4 = \text{constant}$ ), and hence only the last term remains

$$\begin{aligned} &= \sqrt{-G_p} \frac{1}{2} G_{44} G^{44} G_p^{ab} G_{(p)ab,4} \\ &= \sqrt{-G_p} \frac{1}{\sqrt{-G_p}} \partial_{x_4} \sqrt{-G_p} \\ &= \sqrt{-G_p} \partial_{x_4} (4A_G + G_G + 4C_G) \end{aligned} \quad (C.11)$$

where the subscripts are to denote the new frame that we switched to. Switching back to the string frame metric, we find

$$\sqrt{-g_p} e^{-\phi} \partial_{x_4} \left( -\frac{1}{2} \hat{\phi} + G + 2A + 2C \right). \quad (\text{C.12})$$

At this point we stop this analysis because we must also look at the equations of motion coming from the CS action, as this has a cusp as well.

First, for the embeddings that we have chosen, the background form field is the following

$$\begin{aligned} F_{10} &= *M = * \frac{Q_f}{g_s} & A_9 &= \Omega_9 \frac{Q_f}{g_s} e^B |x_4| \\ A_{012356789} &= \sqrt{-g_p} \frac{Q_f}{g_s} e^B |x_4| \end{aligned} \quad (\text{C.13})$$

where  $x_4 \in \{-\pi R_x \cdots \pi R_x\}$ , producing both the positive and negative delta function. In the above, we must use only zeroth order metric functions, as the above statement is already linear in  $Q_f$ . The function  $g_p$  is the determinant of the metric setting  $x_4 = \text{constant}$ , i.e. the pullback metric on the D8s.

Next, we wish to consider the action

$$S_A \equiv \int_{D8} A_9 = \frac{1}{9!} \int d^9 \xi A_{M_1 \dots M_9} \partial_{a_1} X^{M_1} \dots \partial_{a_9} X^{M_9} \epsilon^{a_1 \dots a_9} \quad (\text{C.14})$$

where the epsilon takes values  $\pm 1$ . This is very easy to vary w.r.t. the fields  $X^I$ :

$$\begin{aligned} \frac{\delta S_A}{X^I} &= -\frac{9}{9!} \partial_{a_1} A_{I M_2 \dots M_9} \partial_{a_2} X^{M_2} \dots \partial_{a_9} X^{M_9} \epsilon^{a_1 \dots a_9} \\ &\quad + \frac{1}{9!} \partial_I (A_{M_1 \dots M_9}) \partial_{a_1} X^{M_1} \dots \partial_{a_9} X^{M_9} \epsilon^{a_1 \dots a_9} \end{aligned} \quad (\text{C.15})$$

Taking  $I = i$  to be one of the directions along the world volume coordinates, we see that the first term and second term are identical. This is because in the first equation  $i$  and  $a_1$  must agree to give a non zero answer (when contracting the epsilon). Of course in the sum there are 8! occurrences of this. Hence, the 9! cancels, and one simply gets  $\partial_i A_{0 \dots 9 | x_4}$ , where we use  $|x_4$  to indicate that the index for  $x_4$  has been omitted. The second term is also equal to this, as the contraction yields a (9!). Hence the full equations of motion for the  $X^I$  fields along the volume coordinates vanish.

Next, taking the above equation of motion for the  $x_4$  direction, one finds only a contribution from the second part, i.e.

$$\frac{\delta S_A}{X^{x_4}} = \partial_{x_4} (A_{0 \dots 9 | x_4}) \quad (\text{C.16})$$

Therefore, the full equation of motion for the field  $X^4$  assuming that it is constant reads

$$-\left( \sqrt{-g_p} e^{-\phi} \partial_{x_4} \left( -\frac{1}{2} \hat{\phi} + G + 2A + 2C \right) \pm \partial_{x_4} (A_{0 \dots 9 | x_4}) \right) \quad (\text{C.17})$$

where we have restored a  $-$  sign earlier omitted in front of the DBI action. Also, the  $\pm$  is to be read whether we are putting  $D8$  or  $\bar{D}8$  branes in the background. Both are linear order in  $Q_f$  because to zeroth order none of the metric components depend on  $x_4$ , and  $A_9$  is already linear in  $Q_f$ . Hence, all other metric components are set to being their background values (except those with the  $x_4$  derivative). Considering the cusp solution near  $x_4 = 0$  one reads (factoring out  $Q_f$ )

$$\begin{aligned}
 & -\sqrt{-g_p} \frac{u^{-\frac{3}{4}}}{g_s R_{D4}^{\frac{3}{4}}} \partial_{x_4} \left( \frac{u^{\frac{3}{2}} \left(1 - \frac{U_K^3}{u^3}\right)^{\frac{1}{2}} |x_4|}{(R_{D4})^{\frac{3}{2}}} + \mathcal{O}(x_4^2) \right) \\
 & - \pm \partial_{x_4} \left( \sqrt{-g_p} |x_4| \frac{1}{g_s} \frac{u^{\frac{3}{4}} \left(1 - \frac{U_K^3}{u^3}\right)^{\frac{1}{2}}}{R_{D4}^{\frac{3}{4}}} + \mathcal{O}(x_4^2) \right) \\
 & = -\sqrt{-g_p} \frac{u^{\frac{3}{4}} \left(1 - \frac{U_K^3}{u^3}\right)^{\frac{1}{2}}}{R_{D4}^{\frac{3}{4}} g_s} \partial_{x_4} (|x_4| \pm |x_4|)
 \end{aligned} \tag{C.18}$$

where in the third line, we ignore the higher order in  $x_4$  corrections, as we will evaluate the derivative at  $x_4 = 0$ . Above we have also switched back to the more familiar  $g_s Q_c = 3R_{D4}^3$  notation.

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